Sparse Steiner Triple Systems of Order 21

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Triple Systems

Steiner triple system of order v A collection of 3-subsets (blocks) out of a v-set of points, such that each 2-subset of the v-set occurs in exactly one block, briefly $\mathrm{STS}(v)$

parallel class A set of blocks of an $\mathrm{STS}(v)$ that partition the v points

resolution A partition of the blocks of an $\mathrm{STS}(v)$ into parallel classes

Kirkman triple system of order v An $\mathrm{STS}(v)$ with a specified resolution, briefly $\mathrm{KTS}(v)$

Theorem

There exists an $\mathrm{STS}(v)$ *iff* $v \equiv 1, 3 \pmod 6$.
There exists a $\mathrm{KTS}(v)$ *iff* $v \equiv 3 \pmod 6$.

Example: STS(9)/KTS(9)

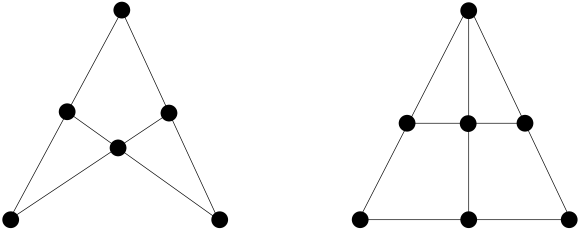
STS(9)	KTS(9)
111100000000	0000
100011100000	0111
100000011100	0222
010001000110	1012
000110001010	1120
001000110010	1201
010000101001	2021
001010000101	2102
000101010001	2210

Configurations in Steiner Triple Systems

(k, l)-configuration A set of l blocks that intersect pairwise in at most one point and whose union contains exactly k points

full configuration A configuration with all points occurring in at least two blocks

1100
1010
1001
0110
0101
0011



Pasch and mitre configuration

Sparse Steiner Triple Systems

r-sparse Steiner triple system A Steiner triple system with no $(l+2, l)$-configuration for any $4 \leq l \leq r$

Conjecture (Erdős)

For every integer r, there exists a value $v_0(r)$ such that there exists an r-sparse $\mathrm{STS}(v)$ for every admissible order $v > v_0(r)$.

$r = 4$: settled
$r = 5, 6$: families are known
$r > 6$: only infinite examples are known

Classification of Steiner Triple Systems

Seeds: (starting points for main search)

1	11111111	00000000	00000000
1	00000000	11111111	00000000
1	00000000	00000000	11111111
0	10000000		
0	10000000		
0	01000000		
0	01000000		
0	00100000		
0	00100000		
0	00010000	**A**	**B**
0	00010000		
0	00001000		
0	00001000		
0	00000100		
0	00000100		
0	00000010		
0	00000010		
0	00000001		
0	00000001		

Perfect Steiner Triple Systems

cycle graph The set of cycles with respect to two points

perfect Steiner triple system An $\mathrm{STS}(v)$ all of whose cycle graphs consist of a $(v-3)$-cycle

Petteri Kaski [Nonexistence of perfect Steiner triple systems of orders 19 and 21, *Bayreuth. Math. Schr.* No. 74 (2005), 130–135]:

- Use approach for classifying Steiner triple systems
- Use a tailored data structure to maintain partial cycle graphs
- Prune the search whenever partial cycle graphs get short cycles

Data Structure for Anti-Pasch Systems

We here consider operations over the binary field $\mathbb{F}_2$.

Observation: Adding the column vectors (of an incidence matrix) that correspond to a Pasch configuration gives the all-zero vector.

Formally: A Pasch configuration corresponds to a word of weight 4 in the dual of the point code of the Steiner triple system.

$\Rightarrow$

Data structure: When adding blocks in the exhaustive search for anti-Pasch Steiner triple systems, maintain all vectors of sums of p columns of the partial incidence matrix for $p = 1, 2, 3$ that have weight 3, 4, and 3, respectively.

Example

111000		111000		00111011110		00000
100110		100110		11001010100		01010
100001		100001		11110001010		10101
010100		010100		10100101001		01001
010001	**1:**	010001	**2:**	10001110010	**3:**	10010
001001		001001		01001101100		01100
001010		001010		01010100011		00011
000100		000100		00100010001		10000
000010		000010		00010000101		00100

Lemma

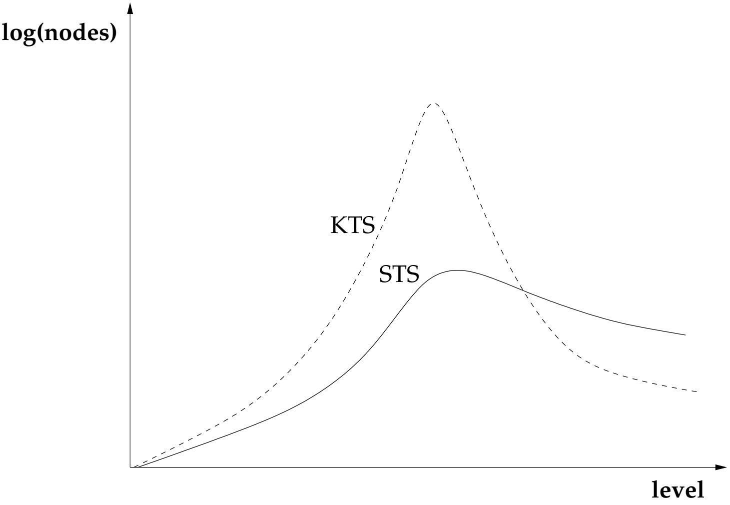
A Pasch configuration in an $\mathrm{STS}(v)$ *constructed from a seed has an even number of blocks in the seed.*

Results

With a cluster of 2.3–2.7-GHz Intel Xeon processors, the anti-Pasch $\mathrm{STS}(21)$ could be classified in approximately 50 core-years. (It took 2 hours to classify the 2591 anti-Pasch $\mathrm{STS}(19)$.) The classification was validated with double counting.

$\lvert\mathrm{Aut}\rvert$	#STS
1	83002911
2	123
3	792
5	24
6	4
7	8
9	3
18	1
21	3
Total	83003869

So Why Are We Interested in These???



Approach for classifying KTS(21)

1. Classify the anti-Pasch KTS(21) from the anti-Pasch STS(21)
2. Classify the KTS(21) with at least one Pasch:

0000000000	**000**0000000	**000**0000000
011**11**11111	**0**1**11**111111	**0**1**11**111111
02222**22**222	**0**222**22**2222	**0**2**2**2**2**22222
1**0**3**1**3**2**3333	1**0**3**12**33333	1**021**333333
2**0**43**1**3**2**444	2**01**33**2**4444	2**01**3**2**44444
33**01**44**2**555	33**01**4**2**5555	33**012**55555
⋮	⋮	⋮

Results so far:

- There are 14 anti-Pasch KTS(21)
- There are 12,520,021 KTS(21) that have a sub-STS(7)

And Why Are the $\mathrm{KTS}(21)$ Interesting???

... except for the fact that this is the instance where combinatorial explosion sets in?

Theorem

There exists a doubly resolvable $\mathrm{STS}(v)$ *iff* $v \equiv 3 \pmod 6$ *with the exceptions of* $v \in \{9, 15\}$ *and a few possible further exceptions:* $v = 21, 141, \ldots$.

Some Final Comments

Proposition

There are three 5-sparse $\mathrm{STS}(21)$ *but no 6-sparse* $\mathrm{STS}(21)$.

Problem

Classify the anti-mitre $\mathrm{STS}(21)$.

Problem

Study switching of Kirkman triple systems.