A localization method in Hamiltonian graph theory

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Joint work with Armen S. Asratian and Nikolay K. Khachatryan

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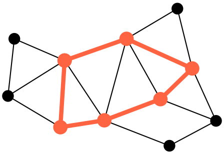
Introduction

Cycles

Cycle:

closed walk $v_1 v_2 \cdots v_n v_1$ with distinct vertices $v_1, \ldots, v_n$



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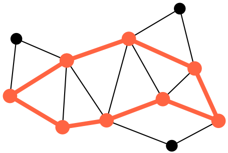
Dominating cycle:

cycle C such that all edges have an endpoint on C



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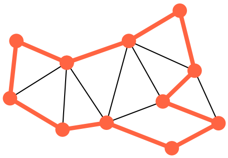
Hamilton cycle: cycle that contains all vertices



Introduction

Dirac (1952)

$|V(G)| \geq 3$, for all x:

$$d(x) \geq |V(G)|/2$$

$\implies$ Hamilton cycle

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Ore (1960)

$|V(G)| \geq 3$, for all non-adjacent x, y:

$$d(x) + d(y) \geq |V(G)|$$

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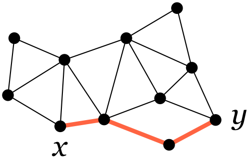
Introduction

Balls

Distance from x to y:

number of edges in shortest x–y-path



Introduction

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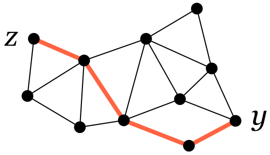
Distance from x to y:

number of edges in shortest x–y-path

Diameter of G:

greatest distance between any two vertices in G



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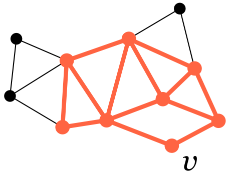
Ball of radius r around v:

part of graph within distance r from v



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$|V(G)| \geq 3$, for all non-adjacent x, y:

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- small diameter ($\mathrm{diam}(G) \leq 2$)
- large edge density ($|E(G)| \geq \text{constant} \cdot |V(G)|^2$)

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Asratian, Khachatryan (1985)

G is connected, $|V(G)| \geq 3$,
for all v and all non-adjacent neighbors x, y of v:

$$d(x) + d(y) \geq |M_2(v)|$$

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A localization method

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A localization method

1. Find equivalent formulation in a ball

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A localization method

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G is connected, $|V(G)| \geq 3$,
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A localization method

1. Find equivalent formulation in a ball
2. Decrease the radius

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1. Find equivalent formulation in a ball
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3. Use the structure of balls

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Asratian, Khachatryan (1990)

G is connected, $|V(G)| \geq 3$,
for all v and all non-adjacent neighbors x, y of v:

$$d(x) + d(y) \geq |N(x) \cup N(y) \cup N(v)|$$

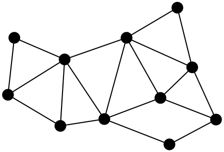
$\implies$ Hamilton cycle

Connectivity

Connectivity

***k*-connected:**

remains connected when deleting $< k$ vertices



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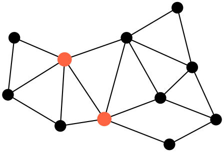

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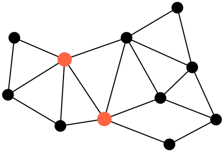
Connectivity, $\kappa(G)$: largest k such that G is k-connected



Localizations of Bondy and Bauer et al.

Bondy (1980)

G is 2-connected, $|V(G)| \geq 3$,
for all non-adjacent x, y, z:

$$d(x) + d(y) + d(z) \geq |V(G)| + 2$$

$\implies$ dominating cycle

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G is connected, all 4-balls are 2-connected, $|V(G)| \geq 3$,
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$\Longrightarrow$ dominating cycle

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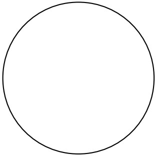

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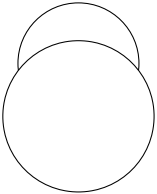

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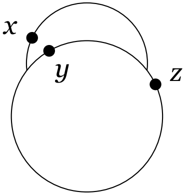

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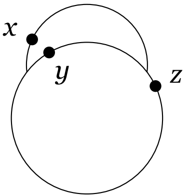

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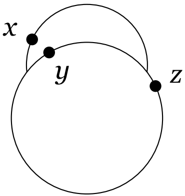

- find x, y, z that *almost* work
- Bondy – partial contradiction
- new x, y, z
- Bondy – full contradiction

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Proof by Wei (1999):

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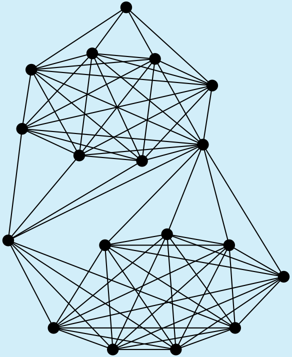

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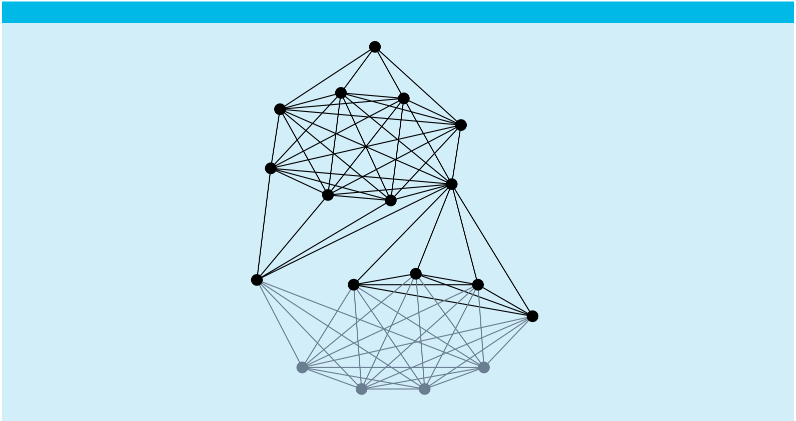

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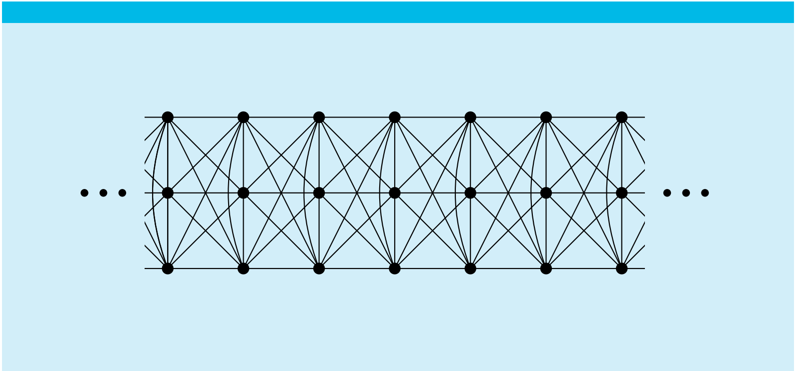
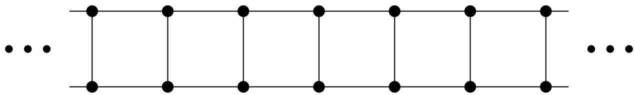


Locally finite infinite graphs

Definitions

Locally finite: vertices have finite degree



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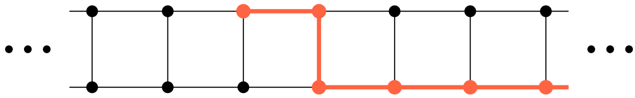
Ray: one-way infinite path



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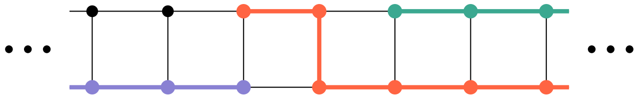
Rays are equivalent if not separated by a finite vertex set



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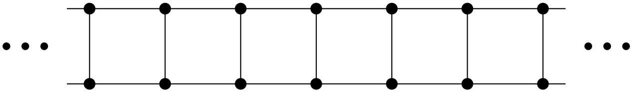
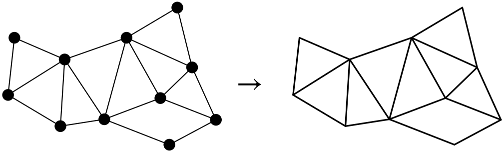
End: equivalence class of the equivalence relation on rays



Locally finite infinite graphs

Freudenthal compactification

graph $G \to$ topological space $|G|$



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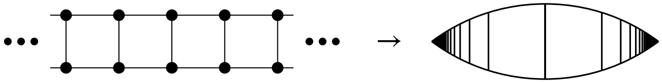
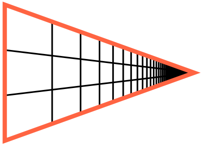


Locally finite infinite graphs

Circles and curves

Circle: embedding of a (topological) circle in $|G|$



Locally finite infinite graphs

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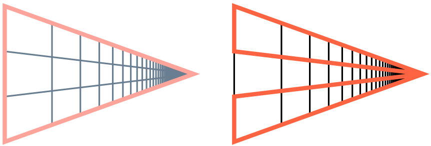
Curve: continuous image of a (topological) circle in $|G|$



Locally finite infinite graphs

Circles and curves

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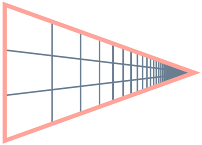
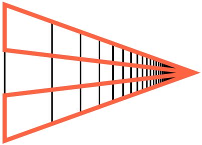
Hamilton circle/curve:
contains all “vertices” without repetition





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Asratian, Granholm, Khachatryan (2018+)

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$$d(x) + d(y) + d(z) \geq |M_3(v)| + \kappa\big(G_3(v)\big)$$

$\implies$ Hamilton curve

Locally finite infinite graphs

Conjecture – Asratian, Granholm, Khachatryan (2018+)

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$\Longrightarrow$ Hamilton circle

Thank you for your attention!

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