On sequential basis replacement in matroids

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- A circuit is a minimal non-independent set.

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- Linear matroid (also known as vectorial or representable)

  Ground set - a set of vectors in a vector space.

  The matroid consists of all the linearly independent sets

The (symmetric) base exchange problem

Gabow (*Mathematical Programming* 1976):

Given two bases A and B of a matroid $\mathcal{M}$, can we exchange their elements one by one, so that after each exchange the resulting sets are bases?

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There is a more general version of this problem by Kajitani and Sugishita (1983)

Some definitions and facts

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- for any $a \in A$ there exists $b \in B$ such that a and b are exchangeable.

The exchange problem in linear matroids

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a_i and b_j are exchangeable if and only if

$M_{ij} \neq 0$ and $M^{-1}_{ji} \neq 0$

An illustration over Z_2

Transition Matrix

0	1	1	0	0	0
1	1	1	1	1	1
1	0	1	1	0	0
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Software by Tal Abziz

https://abziz.github.io/BasisReplacement/#

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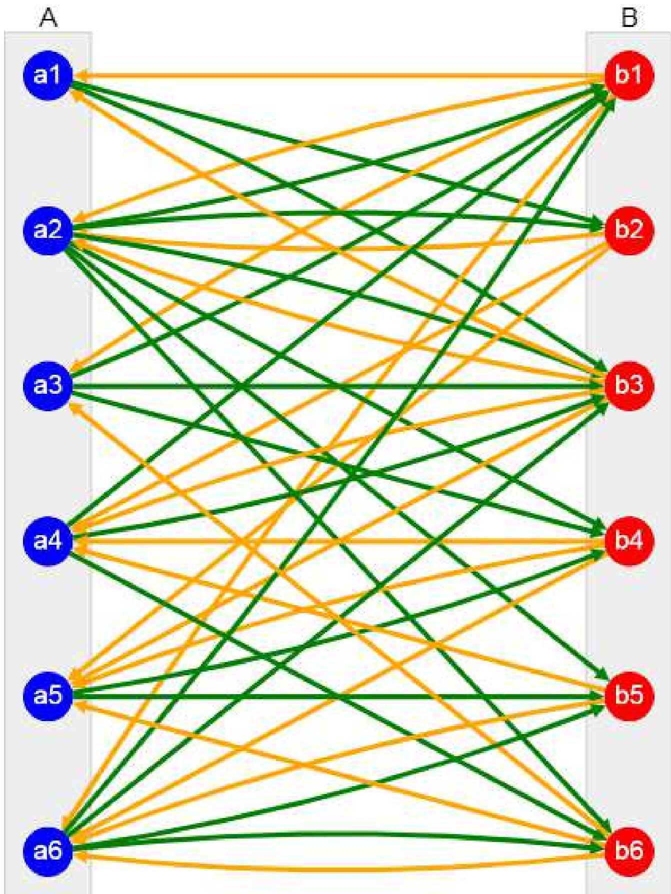

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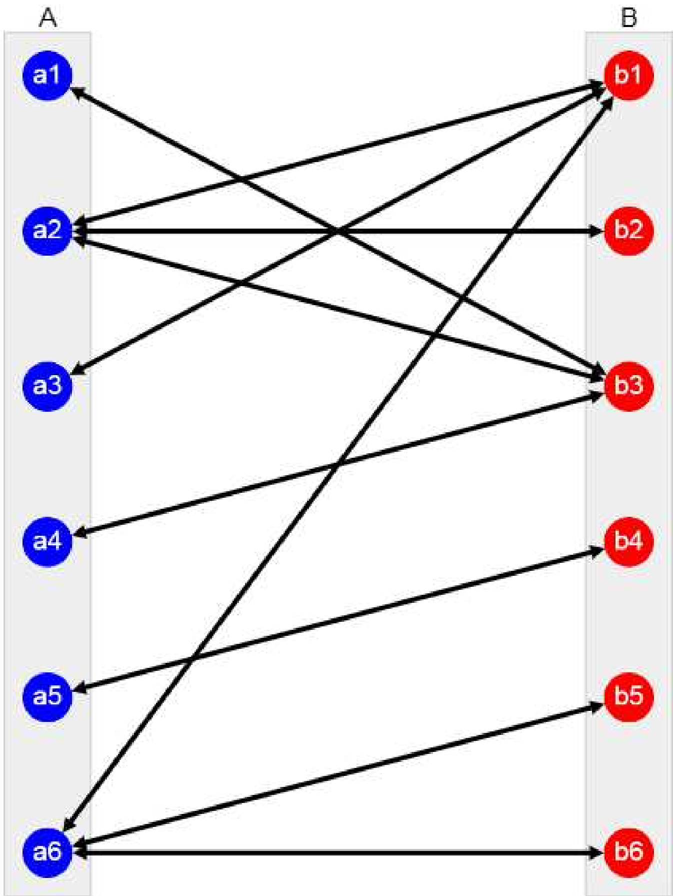

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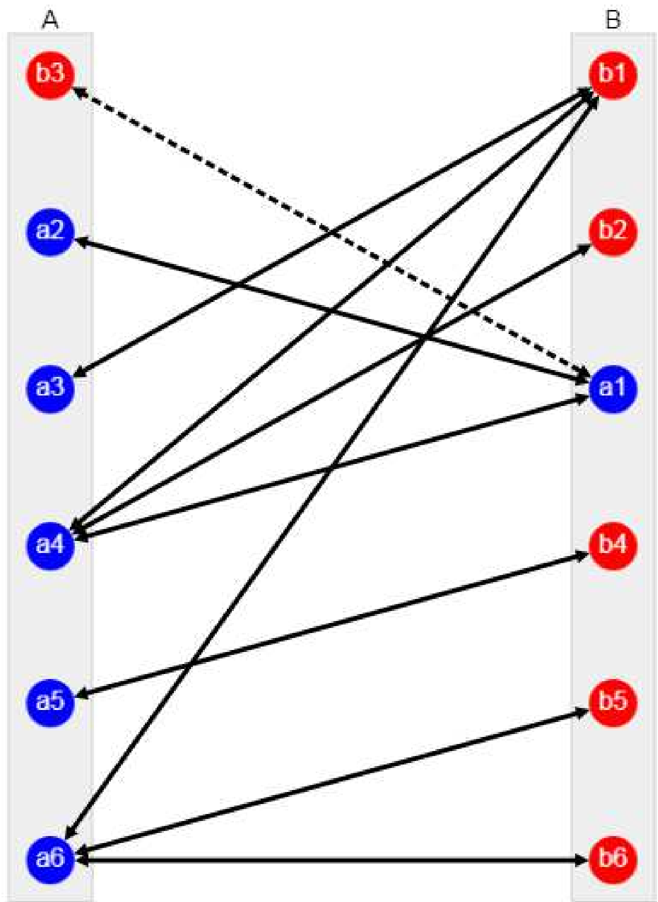

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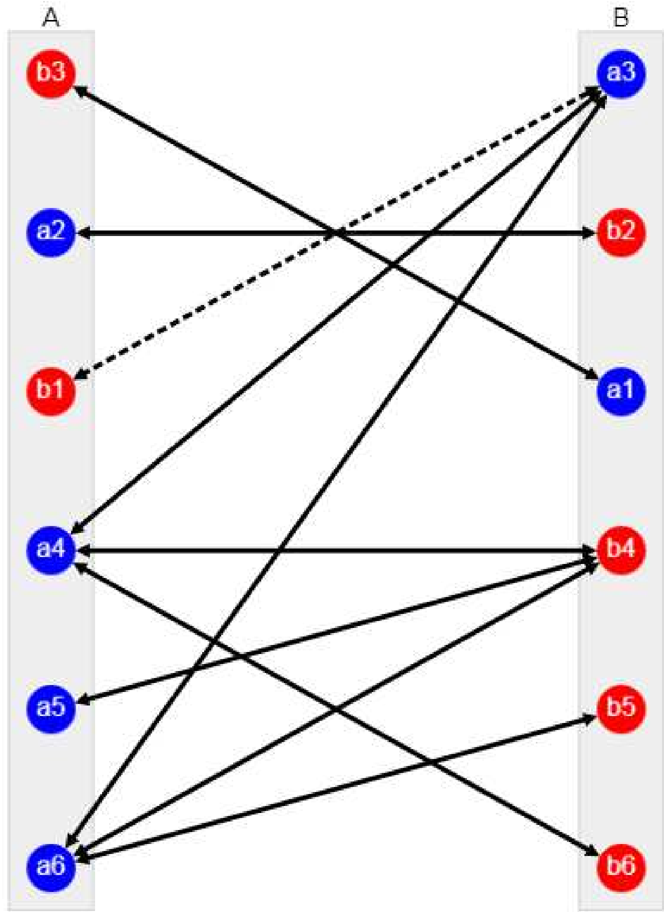

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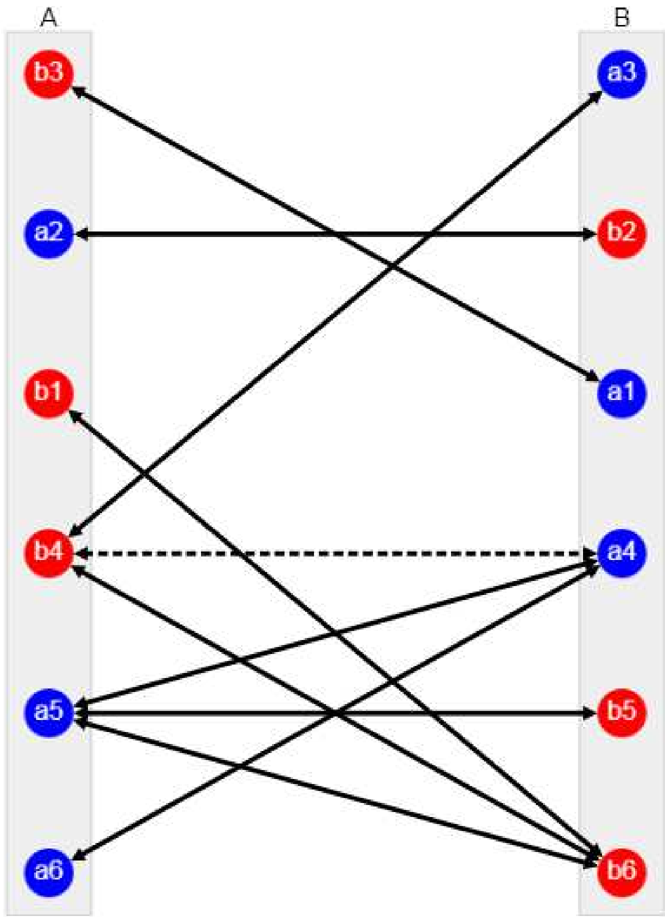

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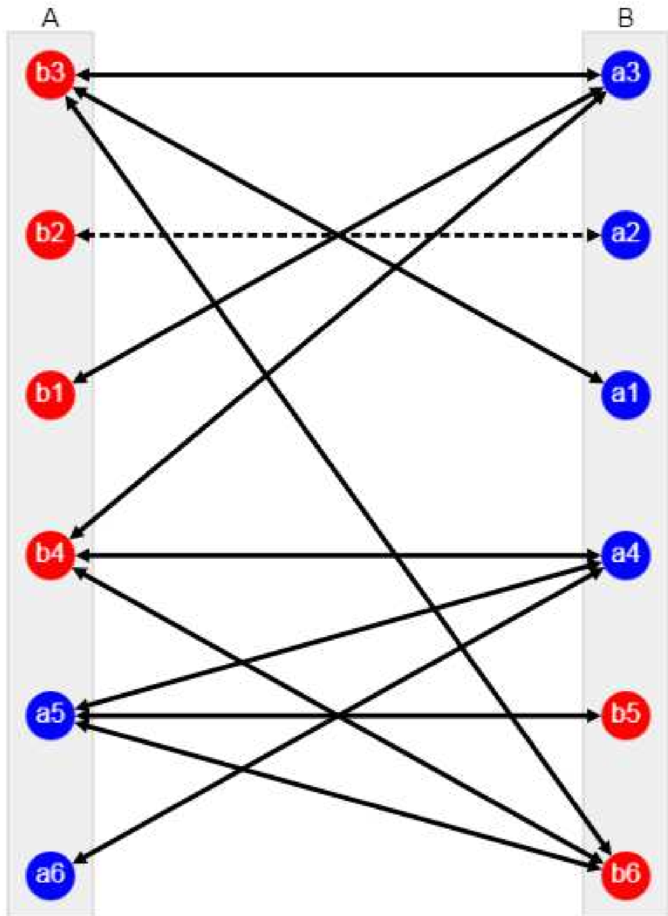

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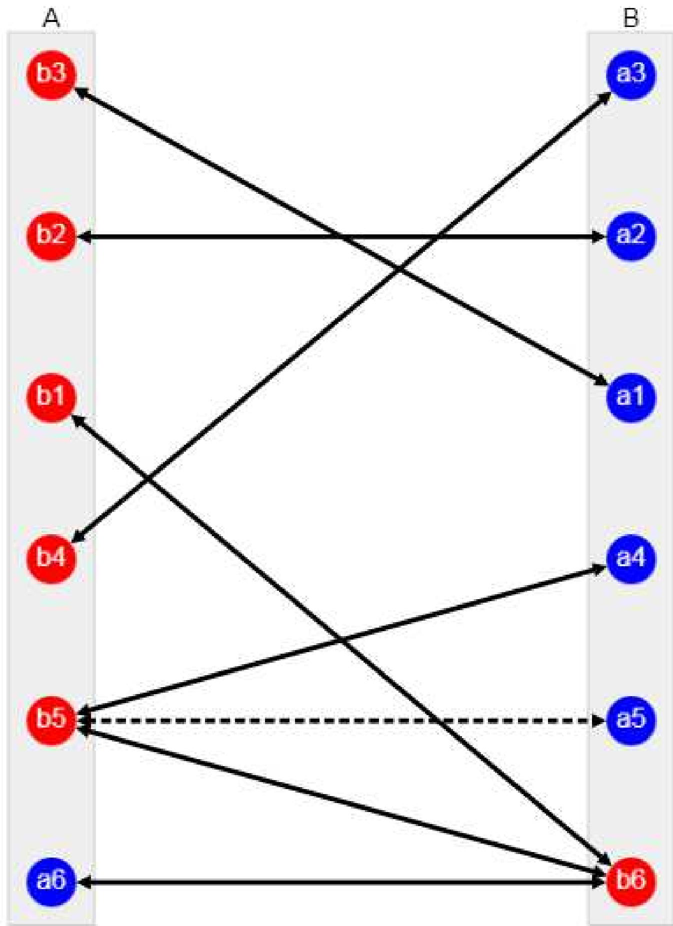

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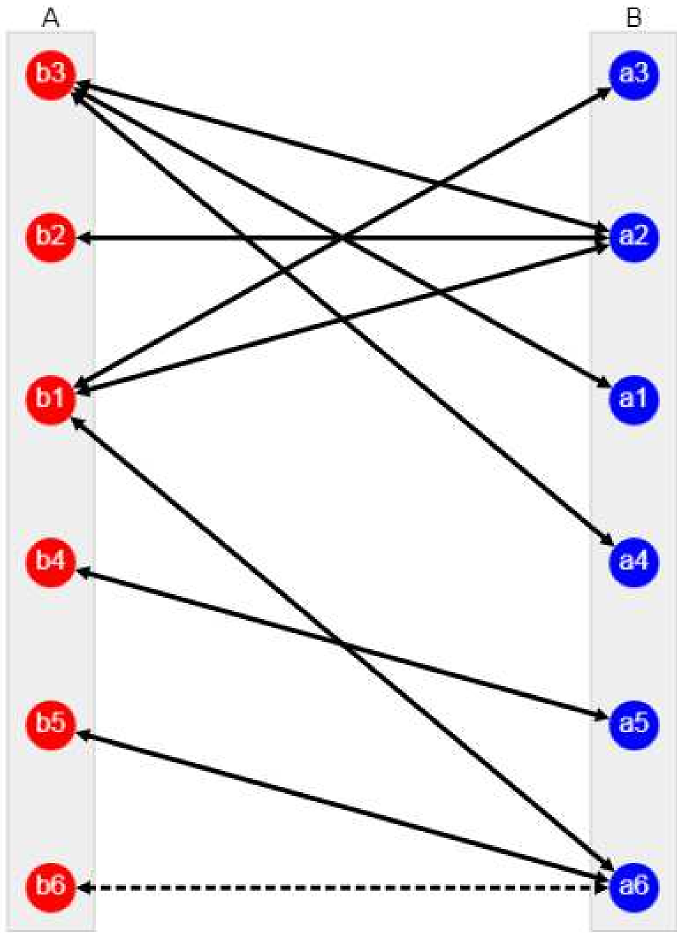

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A problem about determinants

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A positive answer to the basis exchange problem for linear matroids is equivalent to a positive answer for the following problem:

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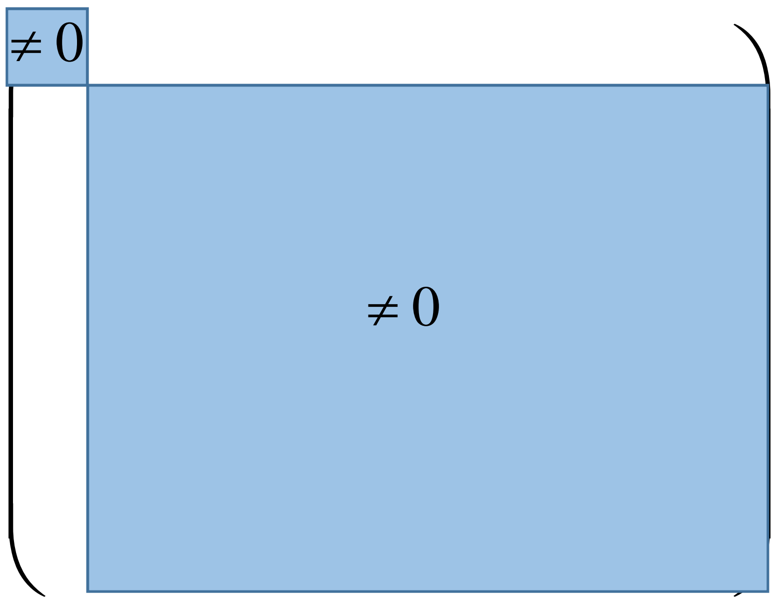
A positive answer to the basis exchange problem for linear matroids is equivalent to a positive answer for the following problem:

Can the rows and columns of any non-singular matrix be rearranged so that all the principal minors and their complements are nonzero?

A problem about determinants

$M = M_{AB} =$



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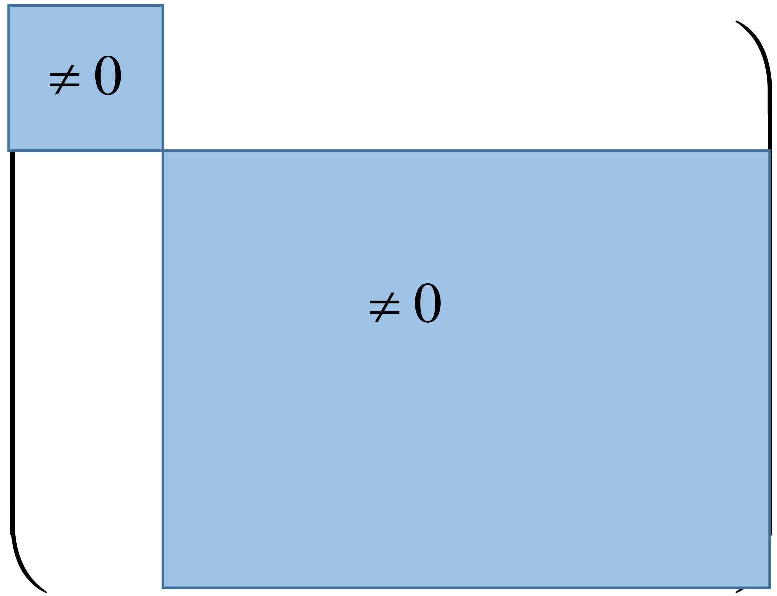

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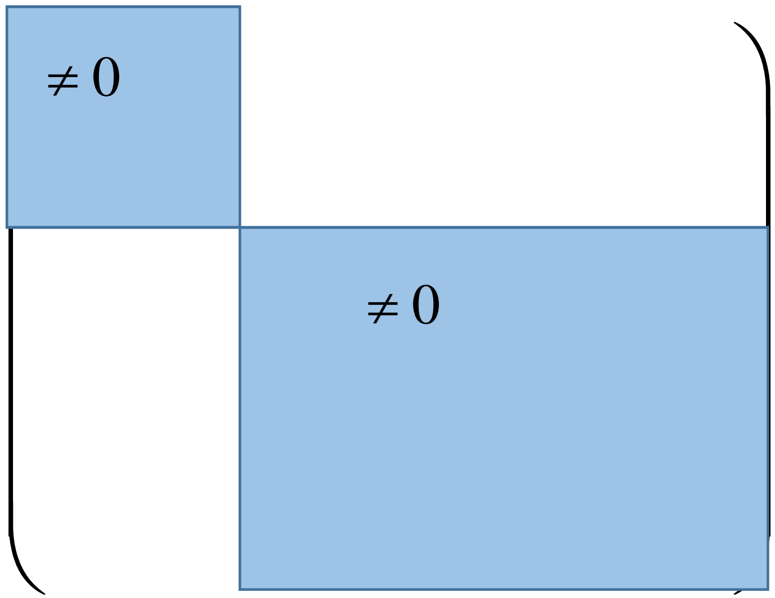

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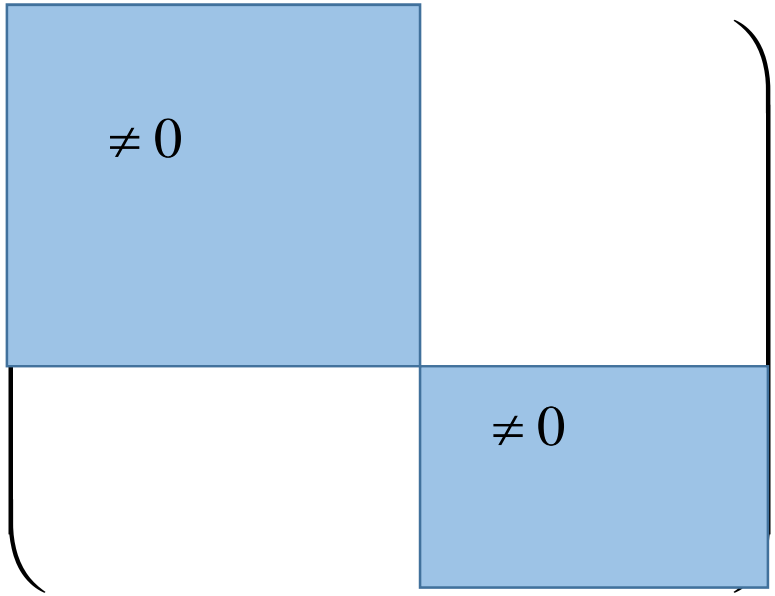

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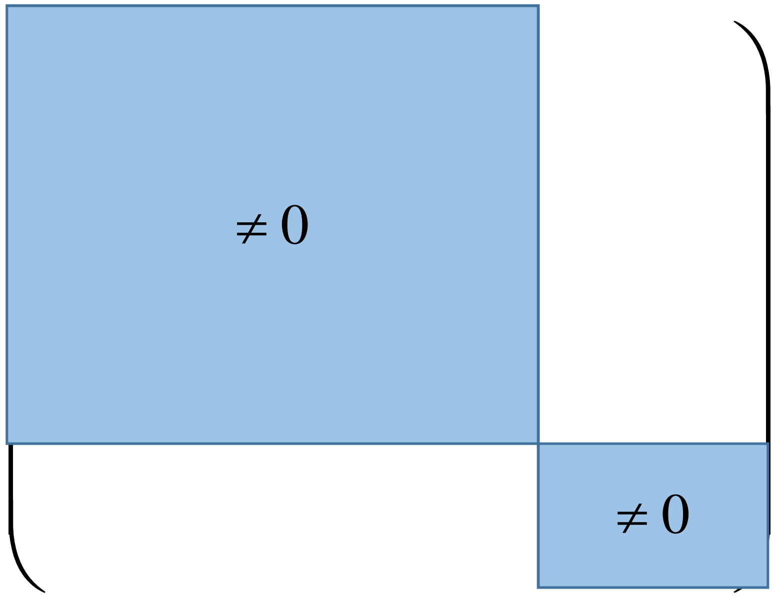
$M = M_{AB} =$  ?

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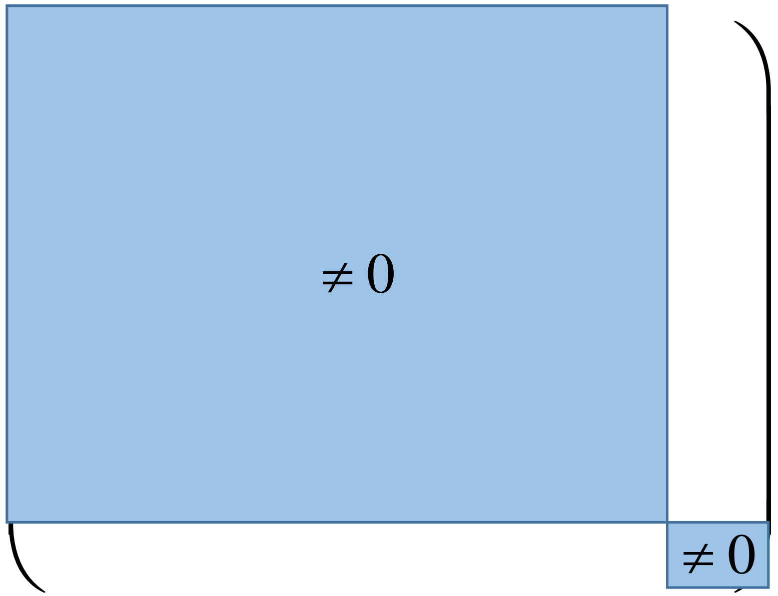

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Towards a solution

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The answer to the base exchange problem is positive for graphic matroids.

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sketch of proof:
Exclude a vertex of minimal degree and apply induction.

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Any two elements of one of the bases can be sequentially exchanged with some two elements of the other basis.

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The answer to the base exchange problem is positive for any matroid of degree at most 5 (but possibly with 6 exchanges).

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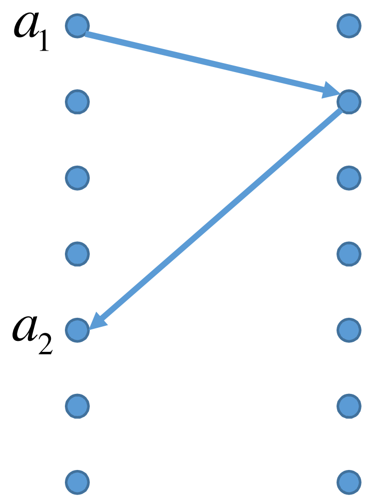

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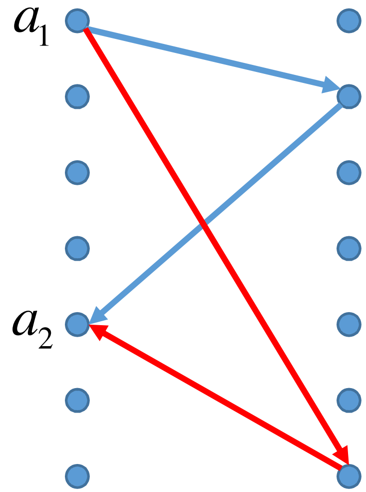

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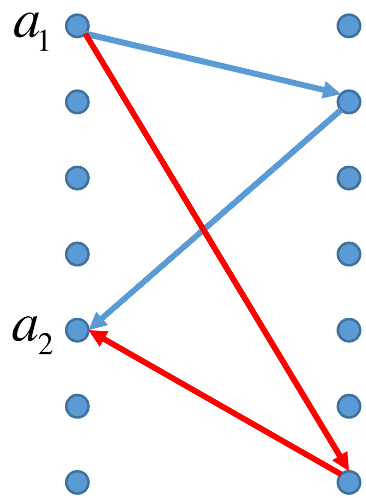

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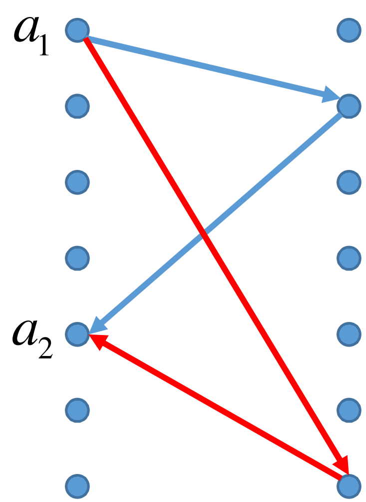

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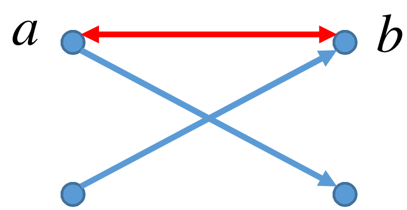

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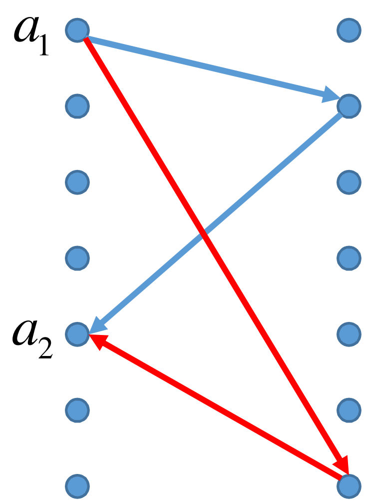

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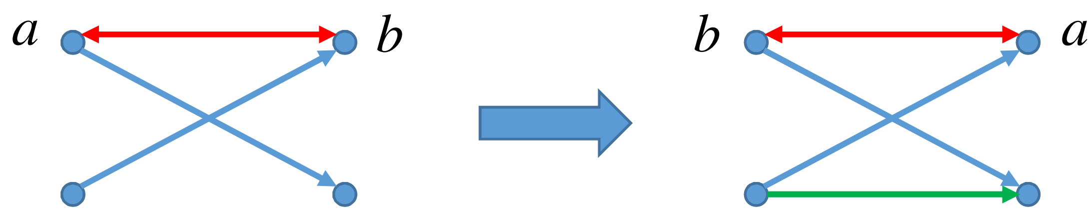

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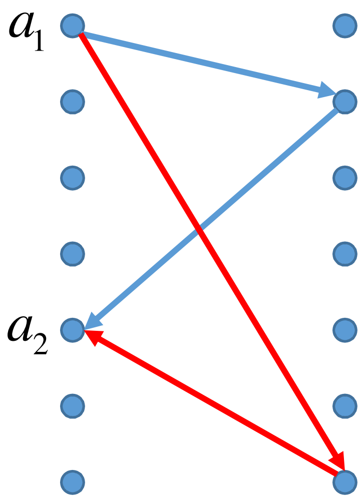

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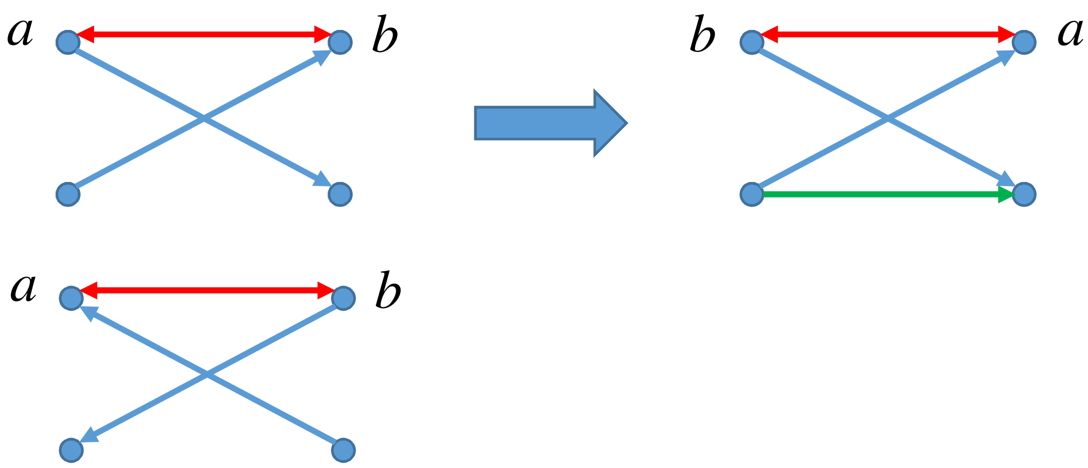

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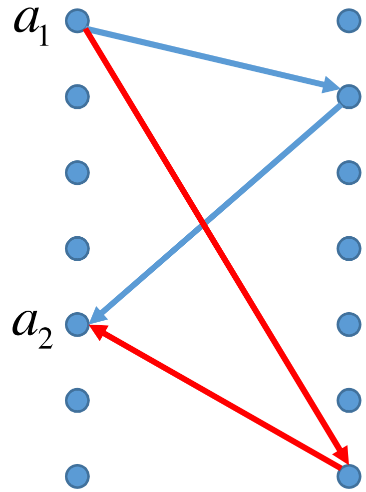

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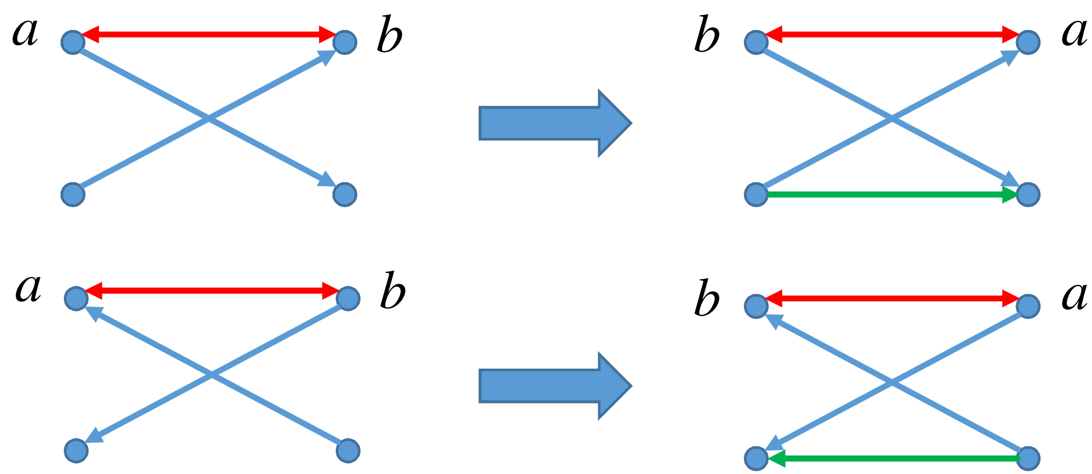

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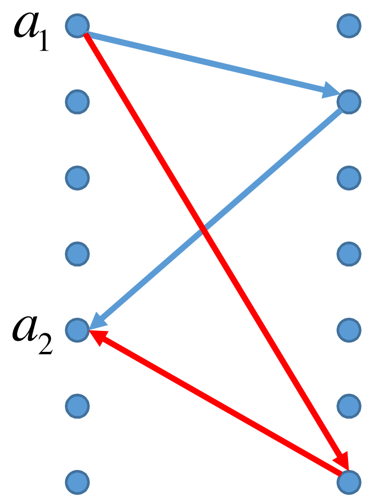

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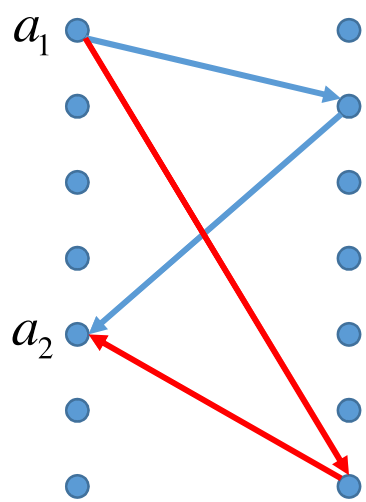

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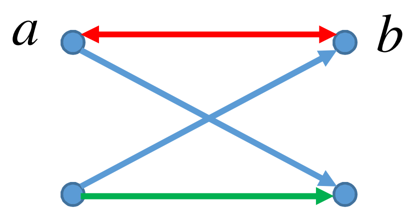

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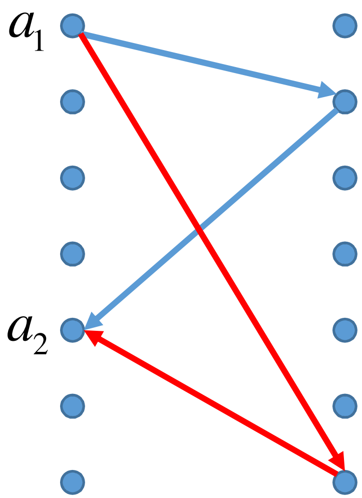

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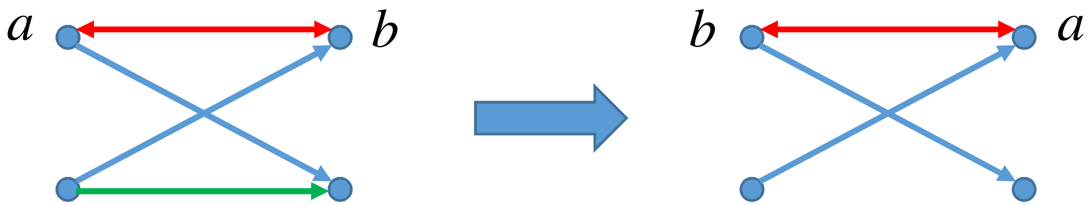

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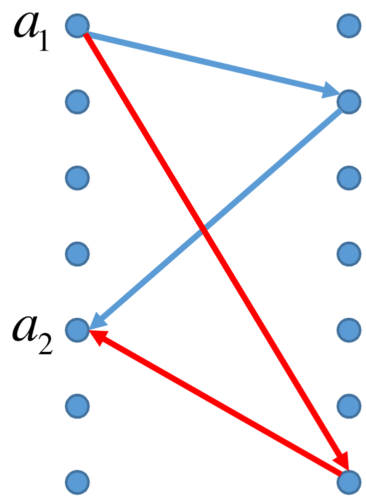

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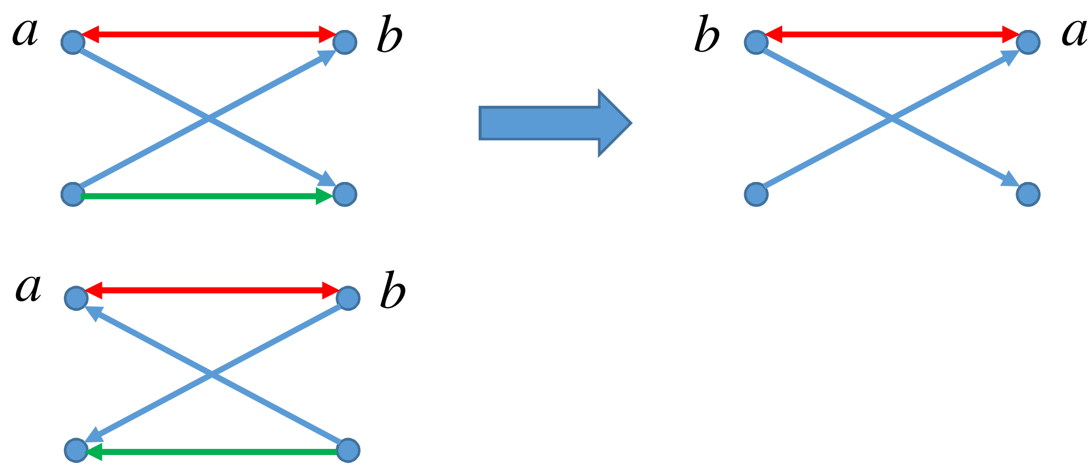

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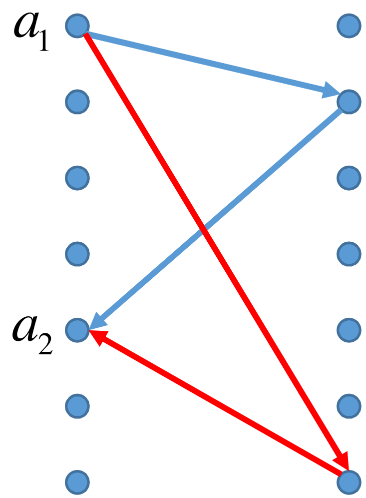

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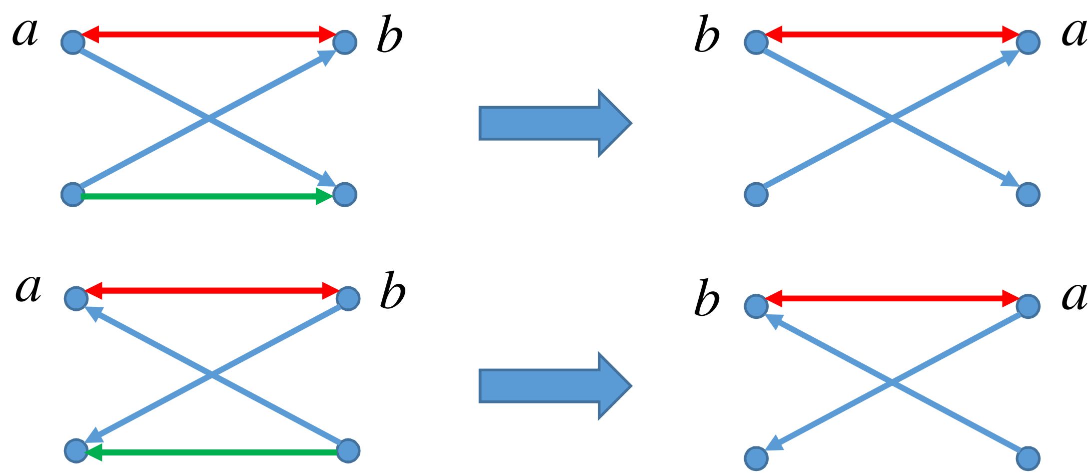

The base-cobase graph problem

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Conjecture 2:

In a block matroid of rank n the diameter of the base-cobase graph is n

The base-cobase graph problem

Cordovil and Moreira (*Combinatorica*, 1993):

Conjecture 2 holds for graphic matroids.

Easing the problem: basis replacement

An easy problem:

Given two bases A and B of a matroid $\mathcal{M}$, can the elements of B, given in a fixed order, replace elements of A, one by one, so that after each replacement the resulting set is a basis?

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A restriction:

$b \in B$ will replace $a \in A$ only if it could have replaced it in the original setup.

Sequential basis replacement

Theorem 1 (K, Roda, Ziv , 2019+)

Let $A=\{a_1,a_2,\ldots,a_n\}$ and $B=\{b_1,b_2,\ldots,b_n\}$ be bases of a matroid $\mathcal{M}$. There exists a permutation σ on $\{1,2,\ldots,n\}$ such that for all $k\in\{1,2,\ldots,n\}$

(i) $A-a_{\sigma(k)}+b_k$ is a basis

(ii) $A-\{a_{\sigma(1)},\ldots,a_{\sigma(k)}\}\cup\{b_1,\ldots,b_k\}$ is a basis

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Two proofs:

1) For the linear matroid case

2) For the general case (using matroid contraction)

The linear matroid case

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The result follows from an easy fact:

The rows of any non-singular matrix can be rearranged so that all the principal minors and all the diagonal elements are nonzero.

A problem about determinants

$$M = M_{AB} = \begin{pmatrix} \neq 0 & & \\ & & \\ & & \end{pmatrix}$$

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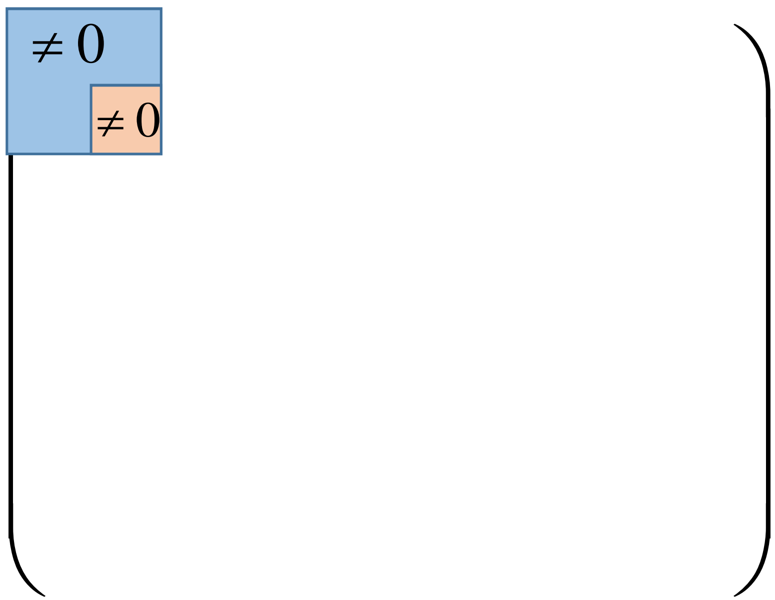

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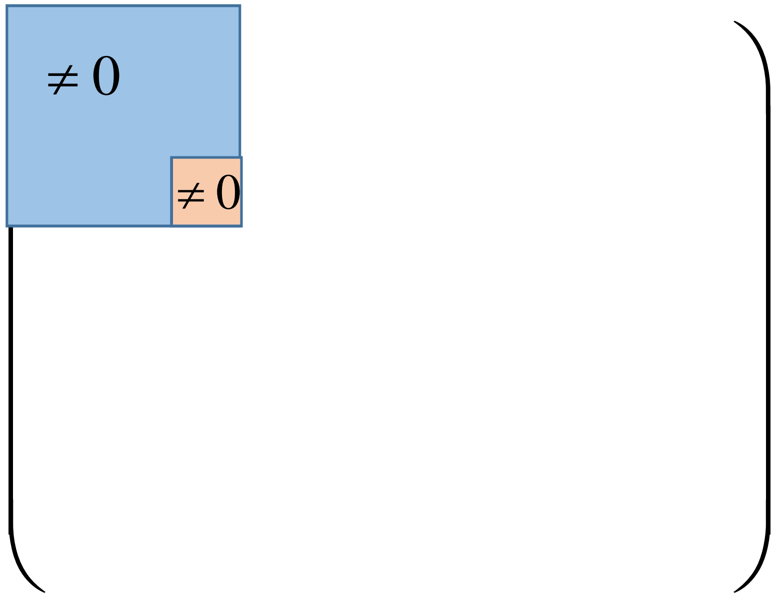

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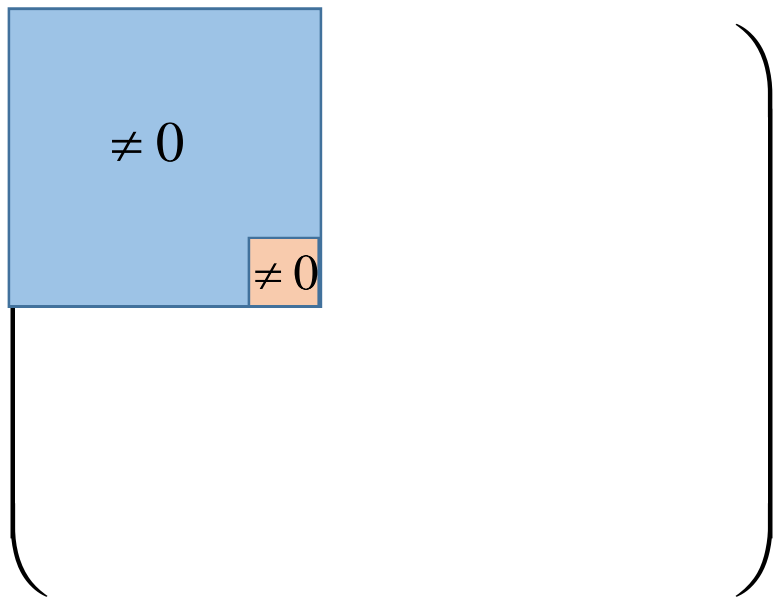

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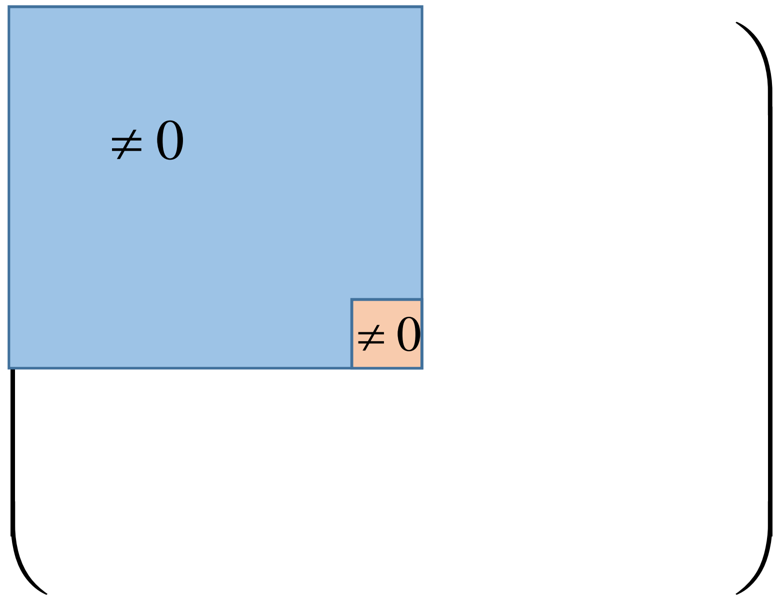

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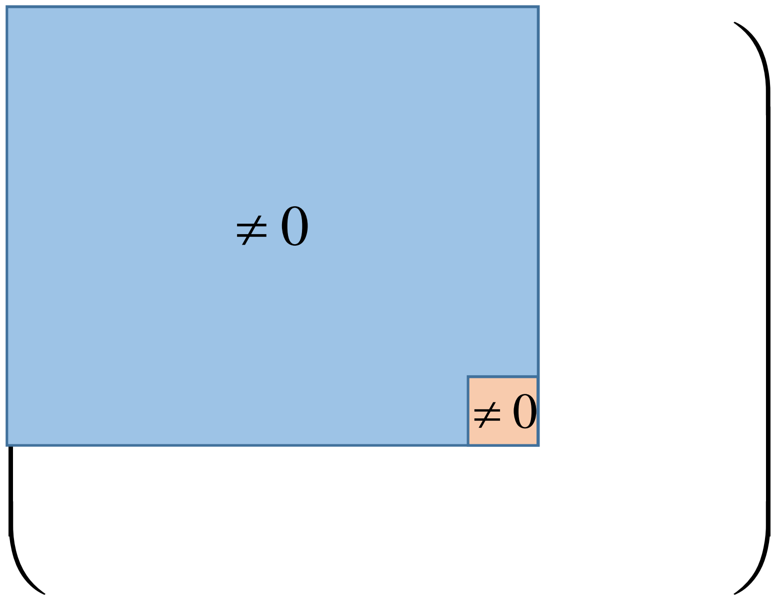

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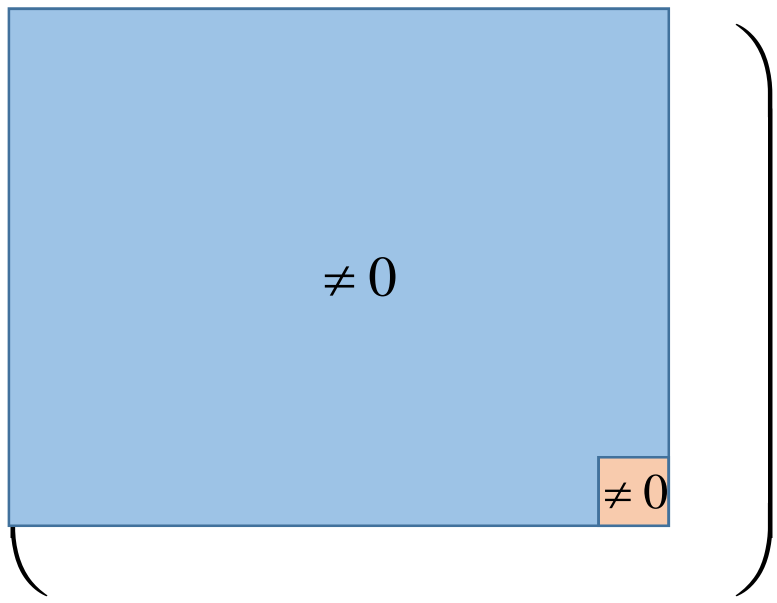

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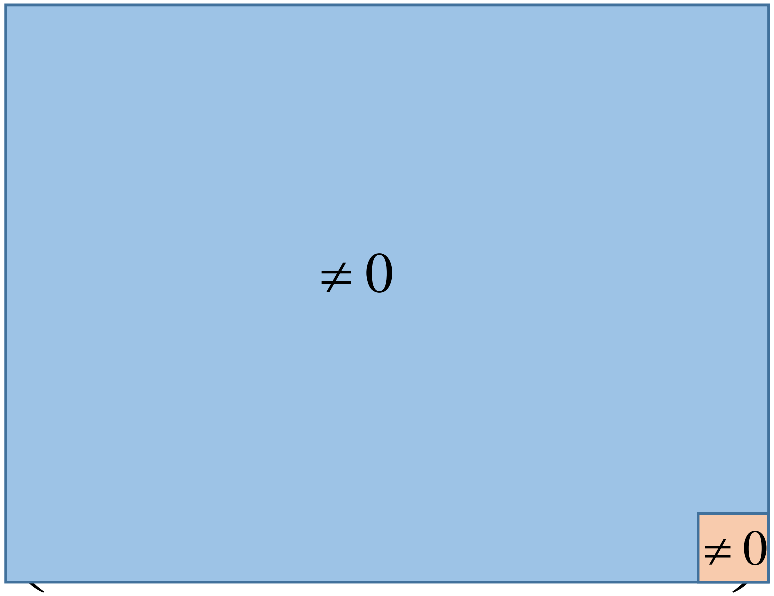

The linear matroid case

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This follows from:

$$\det\left[(a_{ij})_{\substack{1\le i\le n\\1\le j\le n}}\right]\cdot\det\left[(a_{ij})_{\substack{i\ne k,k'\\j\ne l.l'}}\right]=\det\begin{bmatrix}\det\left[(a_{ij})_{\substack{i\ne k\\j\ne l}}\right] & \det\left[(a_{ij})_{\substack{i\ne k\\j\ne l'}}\right]\\ \det\left[(a_{ij})_{\substack{i\ne k'\\j\ne l}}\right] & \det\left[(a_{ij})_{\substack{i\ne k'\\j\ne l'}}\right]\end{bmatrix}$$

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(Dodgson's Condensation Theorem, 1866)

A related result

Observation (Brualdi, 1969):

Given two bases A and B of a matroid $\mathcal{M}$, there is a bijection $\tau : A \to B$ such that $A - a + \tau(a)$ is a basis for all $a \in A$.

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Theorem 1 provides a proof not relying on Hall's theorem

A generalization

Theorem 2 (Donald and Tobey, 1991) :

Given two bases A and B of a matroid $\mathcal{M}$ of rank n, for each $k = 1, \ldots, n$ there is a bijection

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Proof 1 (Donald and Tobey) uses Hall's theorem.

Proof 2 (K, Roda, Ziv, 2019+) w/o Hall's theorem.

Thank you!