Real Zeros and the Alternatingly Increasing Property in Algebraic Combinatorics

Based on joint work with P. Brändén (KTH)

Liam Solus

KTH Royal Institute of Technology

solus@kth.se

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New Property of Interest: Alternatingly Increasing (AI):

$$0 \leq p_0 \leq p_d \leq p_1 \leq p_{d-1} \leq p_2 \leq \cdots \leq p_{\lfloor \frac{d+1}{2} \rfloor}.$$

Symmetric Decompositions:

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Lemma. Let $p \in \mathbb{R}[x]$ be of degree at most d. Then there exist unique polynomials $a, b \in \mathbb{R}[x]$ such that

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- a and b need not have nonnegative coefficients.
- p is AI if and only if a and b both unimodal with nonnegative coefficients.

Questions on AI and symmetric decompositions of polynomials are popping up in algebraic combinatorics!

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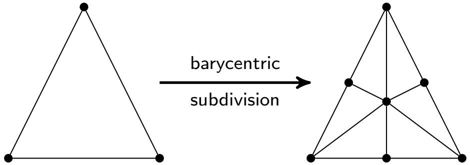

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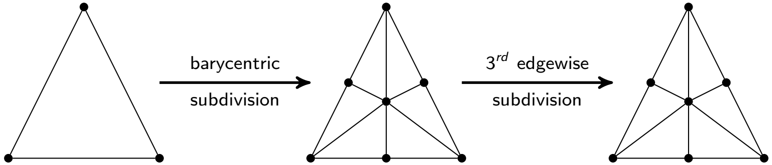

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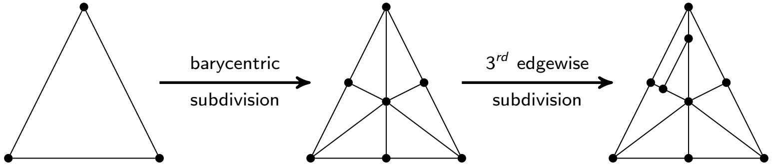

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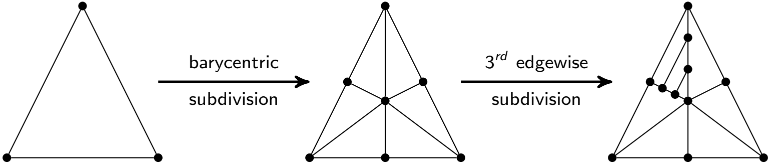

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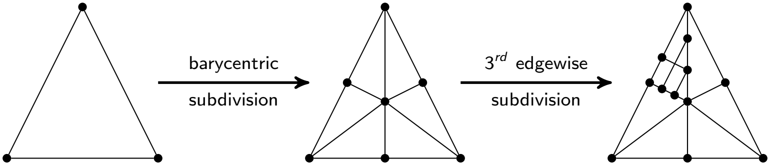

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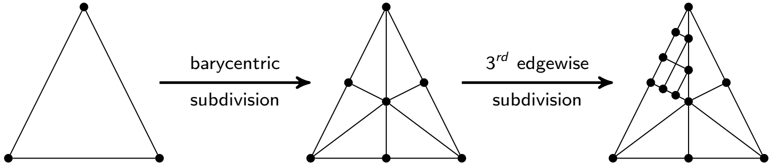

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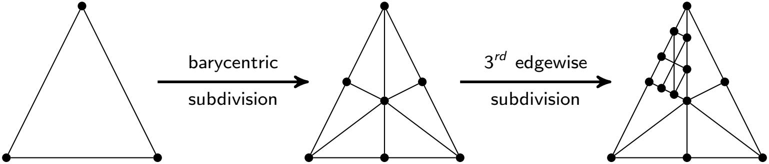

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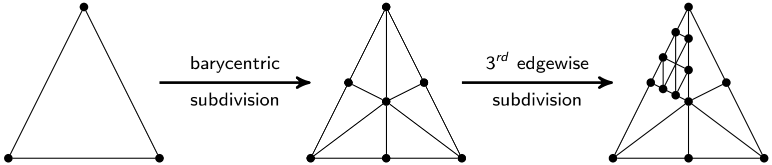

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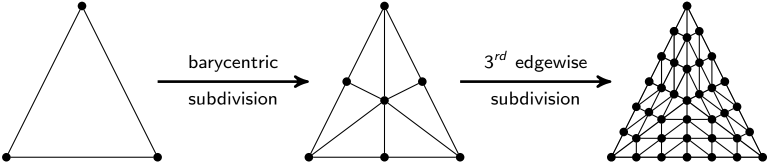

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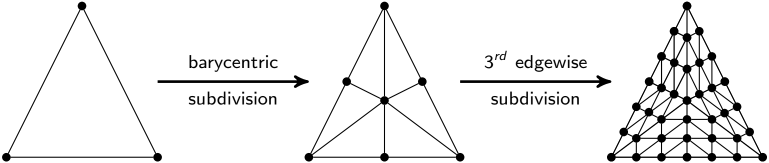

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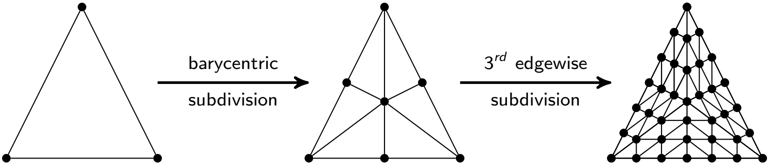

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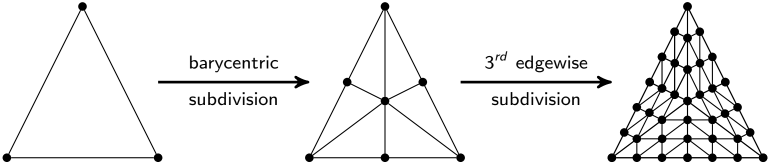

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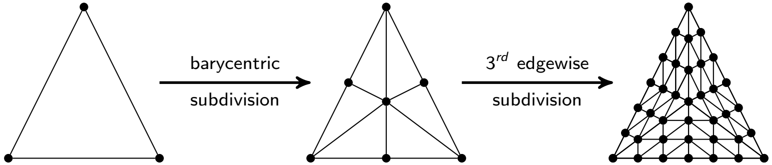

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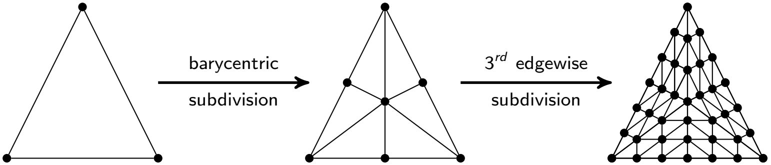

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Question (Athanasiadis). Is $\ell_n(\Omega_{n,r}; x)$ γ-positive?

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$$(\pi, c) \sim \pi_1^{c_1}\pi_2^{c_2}\cdots\pi_n^{c_n},$$

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Theorem (Athanasiadis, 2014). If (a, b) is the symmetric $\mathcal{I}_n$-decomposition of $d_{n,r}$ then a and b are γ-positive and $a = \ell_n(\Omega_{n,r}; x)$.

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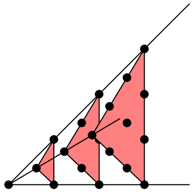

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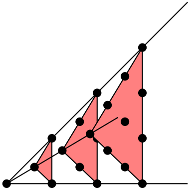

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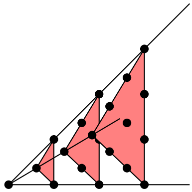

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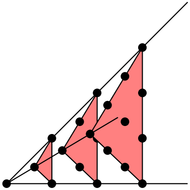

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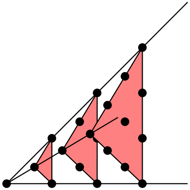

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Conjecture (Stanley, 1989; Hibi and Ohsugi, 2001; etc...). If A_P is integrally closed then $h^*(P; x)$ is unimodal.

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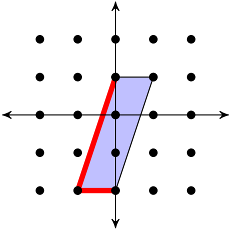

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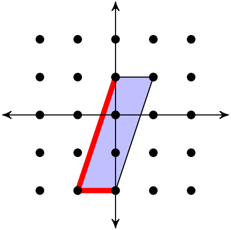

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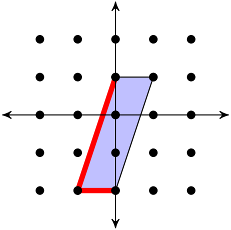

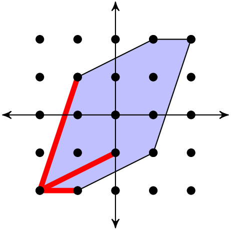

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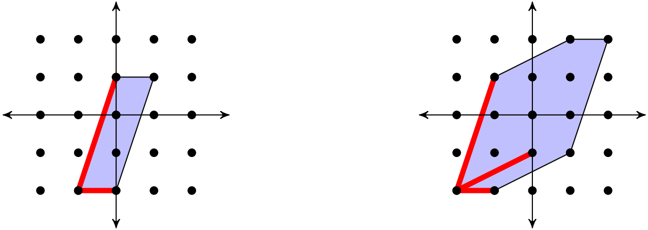

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Conjecture (Beck et al., 2016). True for all lattice zonotopes?

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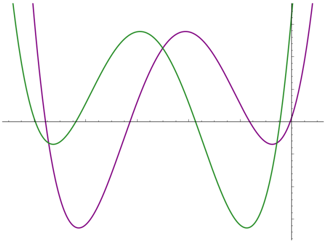

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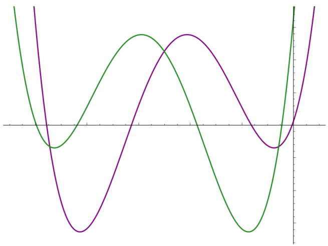

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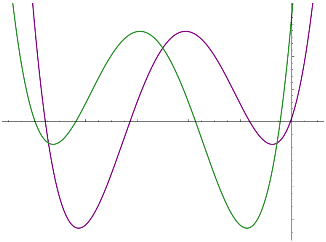

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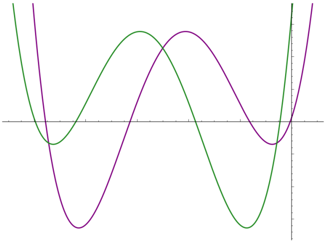

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Corollary. $b \prec a$ for (a, b) the $\mathcal{I}_d$-decomposition of $d_{n,r}$.

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Theorem (Brändén, LS; 2019). Let

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Corollary. If P is a lattice zonotope with interior lattice points then $h^*(P; x)$ is AI. In fact both a and b in the $\mathcal{I}_d$-decomposition of $h^*(P; x)$ are real-rooted.

Thank you for listening!

Please check out our paper:

- P. Brändén and L. Solus. *Symmetric decompositions and real-rootedness.* To appear in International Mathematics Research Notices IMRN (2019).