Quaternary complex Hadamard matrices of order 18

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Introduction

Definition

A *Butson-type Hadamard matrix* with parameters q and n is an $n \times n$ complex matrix, such that $HH^* = nI_n$ and every entry h fulfils $h^q = 1$. The set of all such matrices is denoted $\mathrm{BH}(q, n)$.

Example

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \in \mathrm{BH}(2,4), \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \mathbf{i} & -\mathbf{i} \\ 1 & -1 & -\mathbf{i} & \mathbf{i} \end{bmatrix} \in \mathrm{BH}(4,4).$$

Two matrices are equivalent if one may be gained from the other by a series of row and column permutations and multiplications.

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[Tur70] proposed the construction

$$\mathrm{BH}(4, n) \rightarrow \mathrm{BH}(2, 2n),$$

which gives the following stronger conjecture.

Conjecture

$\mathrm{BH}(4, 2n)$ *is nonempty for all* $n \in \mathbb{N}$.

The classified cases

n	$\mathrm{BH}(2, 4n)$		$\mathrm{BH}(4, 2n)$	
1		1		1
2		1		2
3		1		1
4	[Hal61]	5	[Szö12]	15
5	[Hal65]	3	[LSÖ13]	10
6	[ILL81, Kim89]	60	[LSÖ13]	319
7	[Kim94]	487	[LSÖ13]	752
8	[KTR13]	13 710 027	[LÖS17]	1 786 763
9			New	3 830 723

Table: Numbers of equivalence classes for $\mathrm{BH}(2, 4n)$ and $\mathrm{BH}(4, 2n)$

Types

Definition

Let j_k be a k-dimensional vector of ones. The *type* $t(x, y)$ of two vectors x, y over the 4th roots of unity is the smallest b such that $[x, y]$ is equivalent to

$$\begin{bmatrix} j_a & j_a \\ j_b & \mathbf{i}j_b \\ j_a & -j_a \\ j_b & -\mathbf{i}j_b \end{bmatrix}.$$

Definition

Let $H \in \mathrm{BH}(4, n)$ and denote its columns h_i. Its *column type* is

$$t_{\mathrm{C}}(H) := \min_{i \neq j} t(h_i, h_j).$$

We similarly define the row type $t_{\mathrm{R}}(H)$.
Both types are invariant under equivalence.

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Proof.

Assume $t_{\mathrm{C}}(H) = 0$, and H contains a submatrix equivalent to

$$\begin{bmatrix} j_n & j_n & a \\ j_n & -j_n & b \end{bmatrix}.$$

Hence $j_n^* a = 0$, but an odd-length sum of 4th roots of unity is necessarily nonzero. □

Theorem (Similar to [Kim94])

If n is odd and $H \in \mathrm{BH}(4, 2n)$*, then* $t_{\mathrm{C}}(H) = 1$ *implies* $t_{\mathrm{R}}(H) = 1$*.*

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Proof.

With these assumptions, H is equivalent to $\begin{bmatrix} 1 & 1 & \cdots & 1 & \cdots \\ 1 & -1 & \cdots & x & \cdots \\ j & \mathbf{i}j & \cdots & a & \cdots \\ j & -\mathbf{i}j & \cdots & b & \cdots \end{bmatrix}$, where j is an $(n-1)$-dimensional vector of ones. Hence

$$1 + \mathbf{i} + x(1 - \mathbf{i}) + 2j^{\mathrm{T}} a = 0. \tag{1}$$

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Let α_k denote the number of elements $\mathbf{i}^k$ in a. Then

$$\mathrm{Re}((1)) - \mathrm{Im}((1)) = 2\mathrm{Re}(x) + 2j^{\mathrm{T}}(\mathrm{Re}(a) - \mathrm{Im}(a)) = 0,$$
$$\mathrm{Re}(x) = \alpha_1 + \alpha_2 - \alpha_0 - \alpha_3 = 2(\alpha_1 + \alpha_2) - n - 1.$$

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Therefore $\mathrm{Re}(x) = 0$, so $x = \pm\mathbf{i}$. □

Algorithm

Let $H \in \mathrm{BH}(4, 2n)$. Then H is equivalent to

$$H' := \begin{bmatrix} 1 & 1 & 1 & \cdots \\ 1 & 1 & & \\ \vdots & \vdots & & \\ 1 & 1 & & \\ 1 & & & \\ \vdots & & & \end{bmatrix},$$

where the highlighted block has $n - t_{\mathrm{C}}(H)$ rows. If $t_{\mathrm{C}}(H') \leq t_{\mathrm{R}}(H')$, then H' is *valid*.

We define:

G : full isomorphism group on H,

G' : the subgroup generated by the actions that do not touch the highlighted block of H'.

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4. Classify the 18×18 valid matrices w.r.t. G from $t_{\mathrm{C}}(H) \times 18$ matrices.

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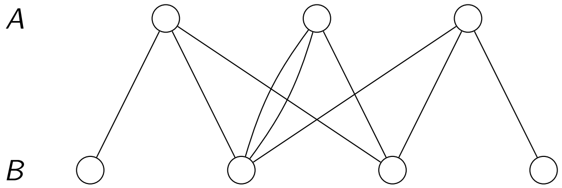

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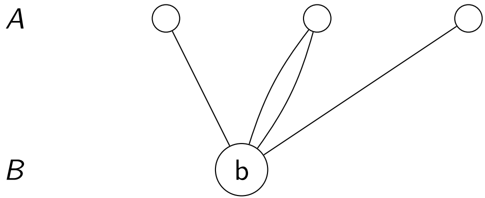

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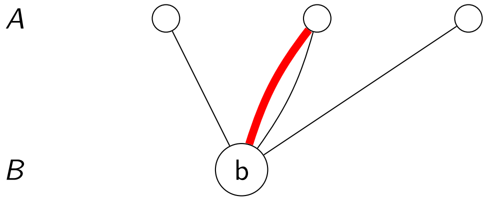

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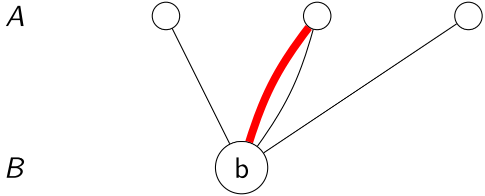
4. Output b if it was extended from the chosen submatrix.



Results

Step	Type			
	1	2	3	4
1	1	1	1	1
2	7	6	4	2
3	628	556	388	86
4	1735156	1641914	984470	0
5	2674809442	2517537560	1205665427	0
6	238649799821	219034416192	74989448690	
7	500814515838	444043909116		
8	7490540949			
18	10425920	64200904	269100	0
Valid	1671122	2157940	137	0
Transposes	0	1524	0	0

The total number of equivalence classes is 3 830 723.

Processing took 18 core years.

On $\mathrm{BH}(2, 36)$

Recall the map $f : \mathrm{BH}(4, n) \to \mathrm{BH}(2, 2n)$ by [Tur70].
By computing $f(\mathrm{BH}(4, 18))$ we found 1 896 248 equivalence classes in $\mathrm{BH}(2, 36)$.
[Orr05] has previously published 18 292 717 $\mathrm{BH}(2, 36)$ matrices, generated from a small set of matrices using switching operations.

Thank you for listening.

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