Automorphisms of an srg(162,21,0,3)

Applications to Mixed Moore Graphs.

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A (*undirected*) **strongly regular graph** with parameters (v, k, λ, μ) satisfies

- it has v verices,
- it is regular of valency k,
- two adjacent vertices have λ common neighbours,
- two non-adjacent vertices have μ common neighbours.

A **Moore graph** is a strongly regular graph with parameters $(k^2+1, k, 0, 1)$.

A Moore graph is a graph M with diameter 2 and girth 5.

Theorem (Hoffman and Singleton 1960)
If a Moore graph M of valency k exists then either

- $k = 2$ and M is C_5,
  (Automorphism group: Dihedral group of order 10)

- $k = 3$ and M is the Petersen graph, P
  (Automorphism group: S_5 of order 120)

- $k = 7$ and M is the Hoffman-Singleton graph,
  (Automorphism group: $P\Sigma U(3,5^2)$ of order 252 000)

or

- $k = 57$ where existence is unknown.
  (Mačaj and Širáň 2010): A Moore graph of valency 57 has at most 375 automorphisms.

A regular mixed graph with

- undirected degree t,
- directed degree z and
- diameter 2

has at most $1+z+t+z(t+z)+t(t+z-1) = (t+z)^2+z+1$ vertices.

If it has exactly $(t+z)^2+z+1$ vertices then it is called a **Mixed Moore graph**.

Special cases:

$z = 0$: undirected, see Hoffman-Singleton

$t = 1$: for every $z \geq 1$ the line-digraph $L(K_{z+2})$ of the complete directed graph is a mixed Moore graph with directed degree z and undirected degree $t = 1$.
Gimbert (2001) proved that these are the unique mixed Moore graphs with $t = 1$.

Feasible cases with $z > 0$ and $t > 1$:

n	t	z	Exists ?
18	3	1	Bosák graph, unique
40	3	3	No, LMF
54	3	4	No, LMF
84	7	2	No, LMF
88	3	6	?
108	3	7	Jørgensen
150	7	5	?
$\vdots$			
486	21	1	?
$\vdots$			

López, Miret, Fernández (2016) proved mixed Moore graphs with 40, 54 or 84 vertices do not exist.
They used a SAT solver.

For a group X and a set $S \subset X$, let Cay(X,S) denote the (directed) **Cayley graph** with vertex set X and edges (x,y) when $yx^{-1} \in S$.

This is an easy way to describe a large graph.
Because it is vertex-transitive it is easy to test properties.

Example. The undirected Moore graphs: the Petersen graph and the Hoffman-Singleton graph are not Cayley graphs.

Example. The mixed Moore graph on 18 vertices: the Bosák graph is a Cayley graph.

Example. The mixed Moore graphs on 108 found by Jørgensen (2015) are Cayley graphs.

Example. The mixed Moore graph $L(K_{z+2})$ is a Cayley graph if and only $z+2$ is a prime power.

Erskine (2017) proved that there are no other mixed Moore Cayley graphs of order at most 485.

Does there exist a mixed Moore (Cayley) graph of order 486.

A mixed Moore graph with $n = 486, t = 21, z = 1$ satisfies:

- The directed edges form 162 triangles.
- Between two triangles there are either no edges or there is a matching.
- The graph obtained by contracting every directed triangle to a vertex is a strongly regular graph srg(162,21,0,3).
- The graph obtained by deleting the directed edges is an antipodal distance regular graph with diameter 4.

Theorem, Makhnev and Nosov 2010
Let g be an automorphism of srg(162,21,0,3) of order 2.
The either

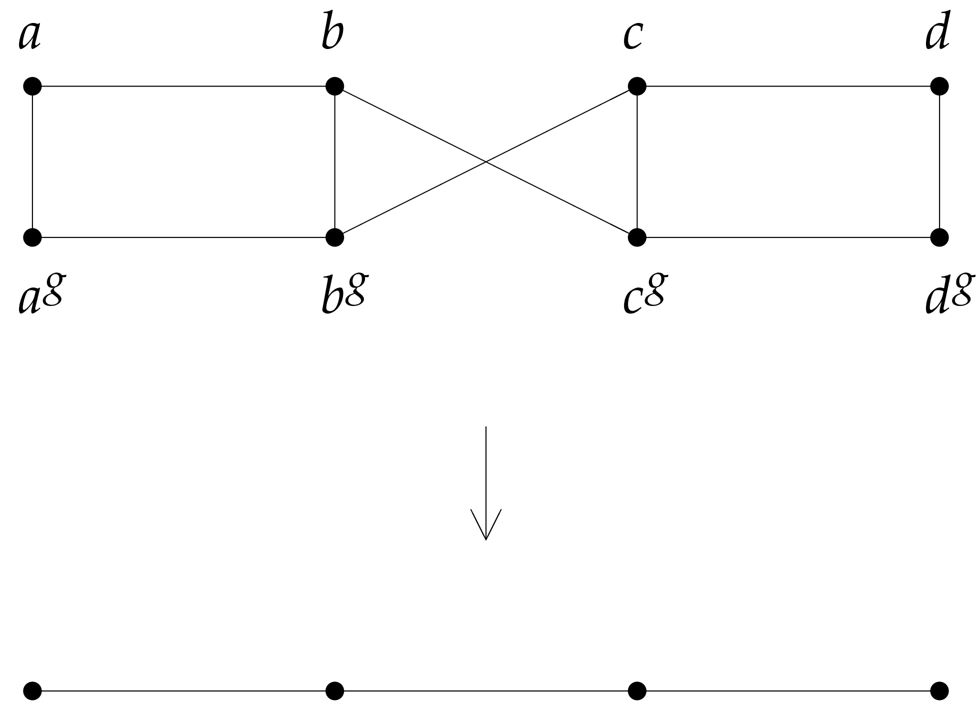
- Fix$(g) = K_{1,3}$ or

- Fix$(g) = \emptyset$ and vertices a and a^g are adjacent.
  Identifying each pair $\{a, a^g\}$ gives the unique srg(81,20,1,6).

Every automorphism of order 2 is an odd permutation.

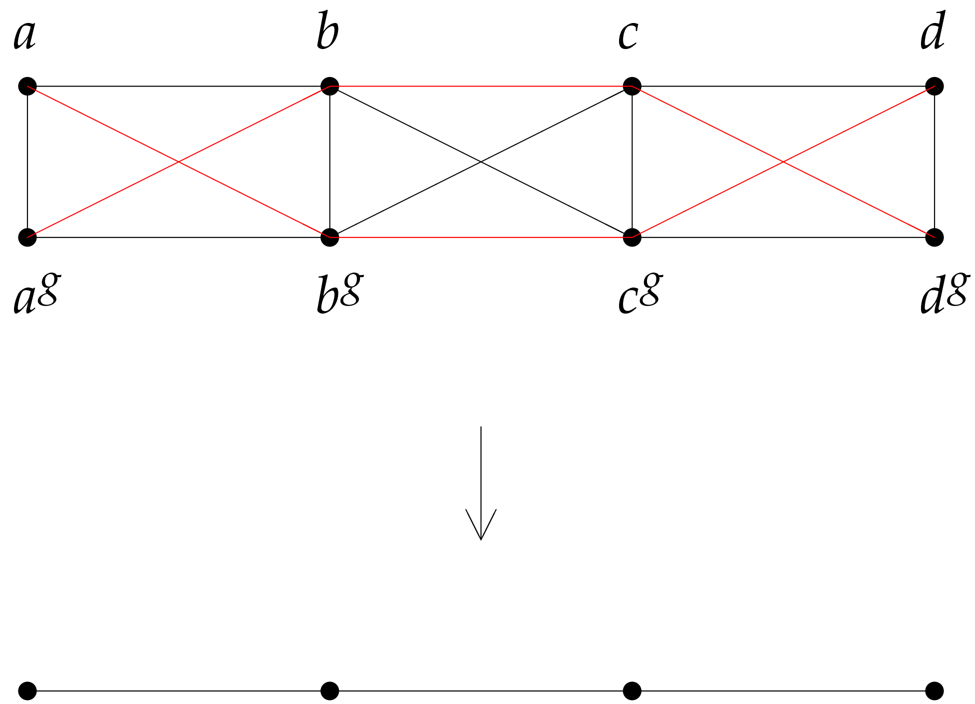
If srg(162,21,0,3) is vertex transitive then every automorphism g of order 2 has Fix$(g) = \emptyset$.

srg(162,21,0,3)



srg(81,20,1,6)

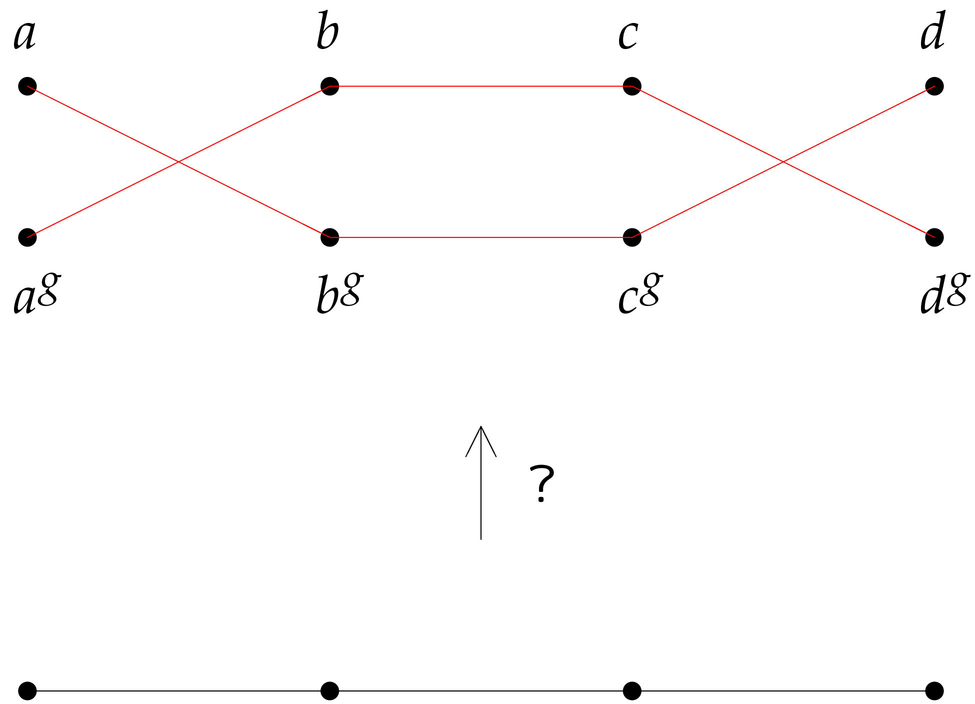
srg(162,21,0,3)



srg(81,20,1,6)

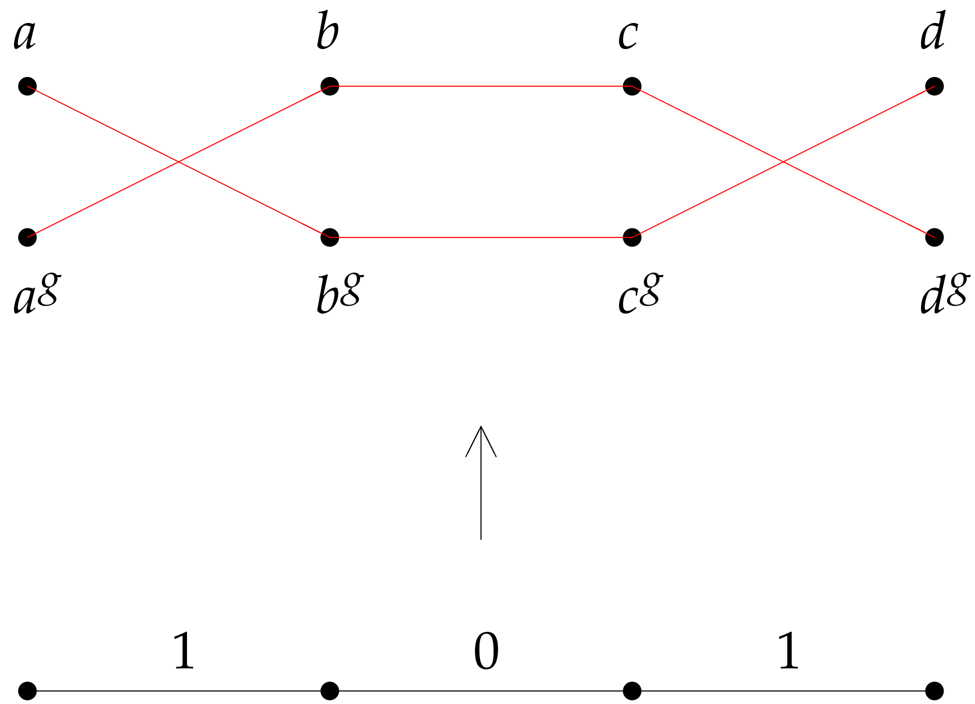
$\{20, 18, 3, 1;\ 1, 3, 18, 20\}$, a 2-cover of srg(81,20,1,6)

2-cover of srg(81,20,1,6)



srg(81,20,1,6) unique Brouwer-Haemers

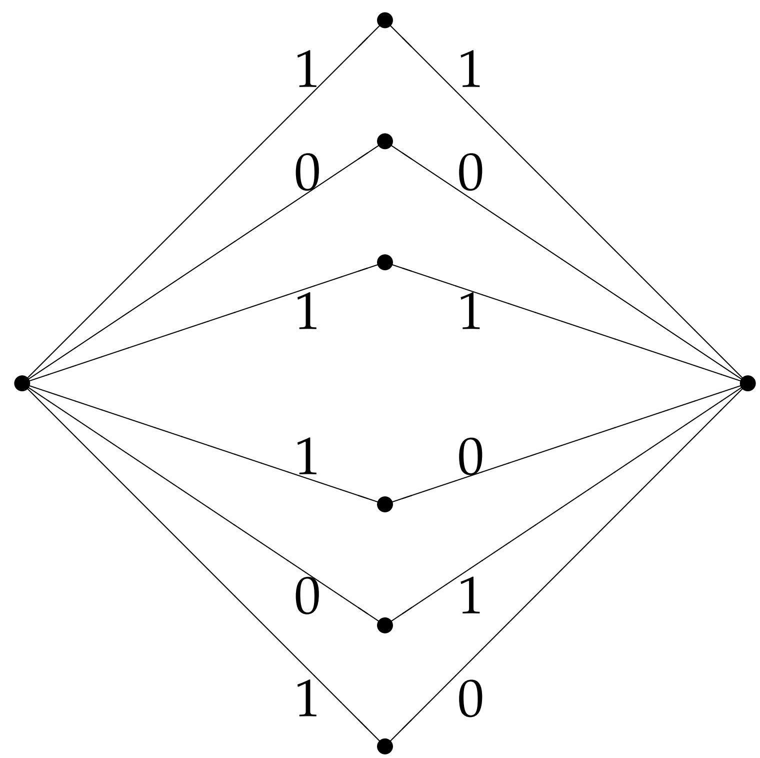
2-cover of srg(81,20,1,6)



srg(81,20,1,6)   unique Brouwer-Haemers

Edges of Brouwer-Haemers graph are labelled 0/1

Condition on labelling of edges in Brouwer-Haemers graph:



For 3 paths: edges of the path have the same label.

Weaker condition:
For 1, 3 or 5 paths: edges of the path have the same label.

Equivalent condition:
Sum of labels of the 12 edges of a $K_{2,6}$ is $\equiv 1 \pmod 2$.

For each of 2430 pairs of non-adjacent vertices in Brouwer-Haemers graph we have
a linear equation over $GF(2)$ in 810 unknowns.

Gaussian elimination:
There are no solutions.

A distance regular graph with intersection array $\{20, 18, 3, 1;\ 1, 3, 18, 20\}$ does not exist.

An srg(162,21,0,3) does not have an automorphism g with order 2 and Fix$(g) = \emptyset$.

An srg(162,21,0,3) can not be vertex transitive.

(Assuming that Makhnev-Nosov theorem is correct:)
A mixed Moore graph of order 486 can not be vertex transitive, as srg(162,21,0,3) is a quotient.

Similarly, the following intersection arrays can be excluded:

1. $\{22, 21, 3, 1;\ 1, 3, 21, 22\}$
   2-cover of srg(100,22,0,6)

2. $\{32, 27, 6, 1;\ 1, 6, 27, 32\}$
   2-cover of srg(105,32,4,12)

3. $\{56, 45, 12, 1;\ 1, 12, 45, 56\}$
   2-cover of srg(162,56,10,24)

Case 2 was excluded by Soicher 2017.
Case 3 was excluded by Brouwer.