Tropical homotopy continuation

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Numerical Alg. Geometry: Homotopy Continuation

Wish to find the roots of

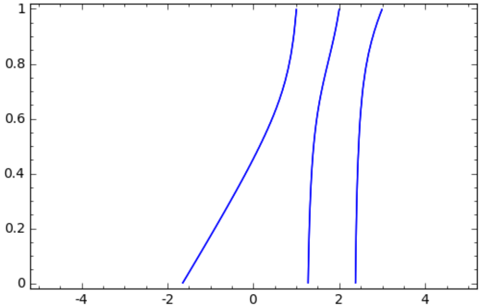
$$f = x^3 - 2x^2 - 3x + 5$$

Know the roots $1, 2, 3$ of

$$g = (x-1)(x-2)(x-3)$$

Form the family of systems

$$H(x,t) = (1-t)(x^3 - 2x^2 - 3x + 5) + t(x-1)(x-2)(x-3).$$



In general one may choose:

$$g_1 := x_1^{\deg(f_1)} - 1$$

$$\vdots$$

$$g_n := x_n^{\deg(f_n)} - 1$$

Called <u>total degree homotopy</u>.

Goal:

a combinatorial/polyhedral version of homotopy continuation.

With this algorithm we can:

- compute mixed volumes of polytopes
- enumerate mixed cells
- provide start systems to numerical homotopy continuation

The algorithm has been implemented in:

- Gfan software for Gröbner bases and polyhedral fans
- Julia package HomotopyContinuation.jl for polynomial system solving by Paul Breiding and Sascha Timme

Tropical Geometry

We evaluate polynomials over $\mathbb{R}$ with operations

- $a \oplus b := \max(a, b)$
- $a \odot b := a + b$

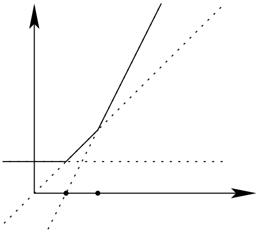
Evaluating $1 \odot x^{\odot 2} \odot y^{\odot 3}$ at $(4, 5)$ gives

$$1 \odot \underbrace{4 \odot 4}_{2} \odot \underbrace{5 \odot 5 \odot 5}_{3} = 1 + 2 \cdot 4 + 3 \cdot 5 = 1 + \begin{pmatrix} 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Tropical polynomials are piece-wise linear.

$$(-2 \odot x^{\odot 2}) \oplus (0 \odot x) \oplus (1)$$

$$= \max(2x - 2, x, 1)$$



For $f \in \mathbb{R}[x_1, \ldots, x_n]$: The "tropical hypersurface" is defined as

$$T(f) := \{(a_1, \ldots, a_n) \in \mathbb{R}^n : \text{ max in } f(a) \text{ is attained} \geq \text{twice}\}$$

Tropical hypersurfaces

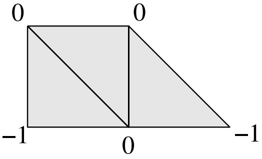
Equivalently, if f has exponent vectors being columns of $A \in \mathbb{Z}^{n \times m}$ and coefficients $c \in \mathbb{R}^m$ then

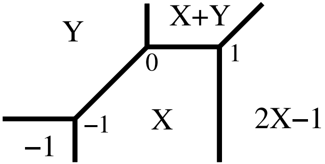
$$f(\omega) = \max_{i=1}^{m}(c_i + \langle a_i, \omega \rangle)$$
$$T(f) = \{\omega \in \mathbb{R}^n : \max_{i=1}^{m}(c_i + \langle a_i, \omega \rangle) \text{ is attained} \geq \text{twice}\}$$

Example

$f = (-1) \oplus (y) \oplus (x) \oplus (x \odot y) \oplus ((-1) \odot x \odot x)$

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}, \; c = (-1, 0, 0, 0, -1).$$





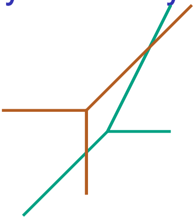
Tropical hypersurfaces

$$T(x_1 \oplus x_2 \oplus x_3 \oplus 0) =$$

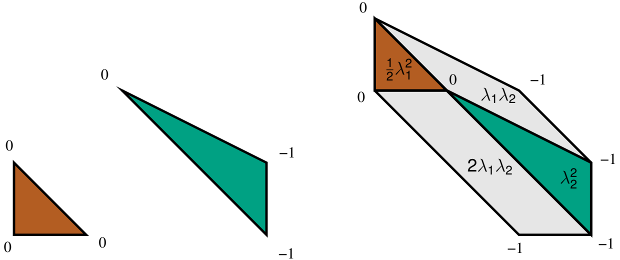
How do solutions to a tropical polynomial system look?

$$f_1 = 0 \odot x \oplus 0 \odot y \oplus 0$$

$$f_2 = (-1) \odot x^{\odot 2} \odot y \oplus (-1) \odot x^{\odot 2} \oplus 0 \odot y^{\odot 2}$$



The intersection points are dual to the mixed cells in the subdivision of $\mathrm{New}(f_1) + \mathrm{New}(f_2)$ using the lift $(0, 0, 0, -1, -1, 0)$



$$VolPol = \frac{1}{2}\lambda_1^2 + \boxed{(1+2)}\lambda_1\lambda_2 + 1\lambda_2^2$$

Mixed volumes

Definition

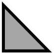
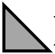
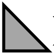
Let $C_1, C_2, \ldots, C_n \subseteq \mathbb{R}^n$ be bounded convex sets. The function

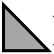
$$f : \mathbb{R}^n \to \mathbb{R}$$

$$(\lambda_1, \ldots, \lambda_n) \mapsto \text{Volume}(\lambda_1 C_1 + \cdots + \lambda_n C_n)$$

is polynomial in the variables $\lambda_1, \ldots, \lambda_n$. The coefficient of $\lambda_1 \cdots \lambda_n$ is called the mixed volume of $C_1, \ldots, C_n$.

Example

Vol($\lambda_1 \cdot$  $+ \lambda_2 \cdot$ ) = Vol($(\lambda_1 + \lambda_2) \cdot$ ) =

$(\lambda_1 + \lambda_2)^2$Vol() $= \frac{1}{2}(\lambda_1 + \lambda_2)^2 = \frac{1}{2}\lambda_1^2 + \boxed{1}\lambda_1\lambda_2 + \frac{1}{2}\lambda_2^2$

Theorem (Bernstein,Kusnirenko,Khovanskii 1975)

If $f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$ are generic then
$|V(\langle f_1, \ldots, f_n \rangle) \cap (\mathbb{C} \setminus \{0\})^n| = \text{MixVol}(New(f_1), \ldots, New(f_n)).$

Specifications

We have a numerical algorithm with these specifications:

Algorithm (Numerical homotopy continuation)

Input: $g_1, \ldots, g_n \in \mathbb{C}[x_1, \ldots, x_n]$ *generic start system*
All solutions $V(g_1) \cap \cdots \cap V(g_n) \subseteq \mathbb{C}^n$
$f_1, \ldots, f_n \in \mathbb{C}[x_1, \ldots, x_n]$ *target system.*

Output: *All* ***isolated*** *solutions in* $V(f_1) \cap \cdots \cap V(f_n)$

We want a tropical (combinatorial) algorithm:

Algorithm (Tropical homotopy continuation)

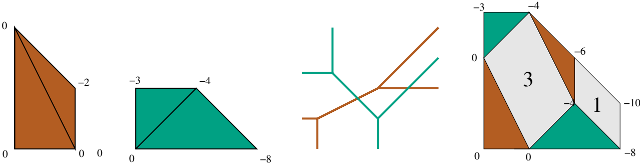
Input: $g_1, \ldots, g_n \in \mathbb{R}[x_1, \ldots, x_n]$ *generic start system*
The finite set $T(g_1) \cap \cdots \cap T(g_n) \subseteq \mathbb{R}^n$
$f_1, \ldots, f_n \in \mathbb{R}[x_1, \ldots, x_n]$ *target system.*

Output: *All* ***isolated*** *solutions in* $T(f_1) \cap \cdots \cap T(f_n)$

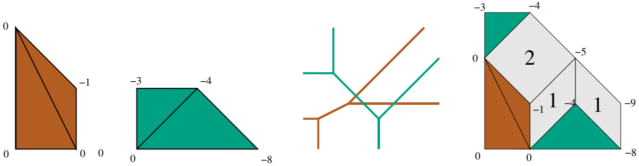
How solutions move around as coefficients change

$$A_1 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

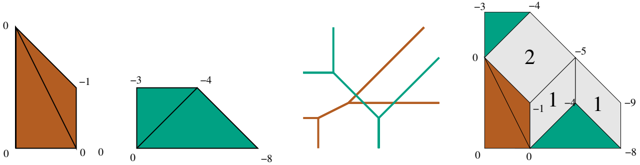
Choosing $w_1 = (0, 0, 0, -2)^t$ and $w_2 = (0, -3, -4, -8)^t$ we get the two tropical hypersurfaces shown in the middle picture.



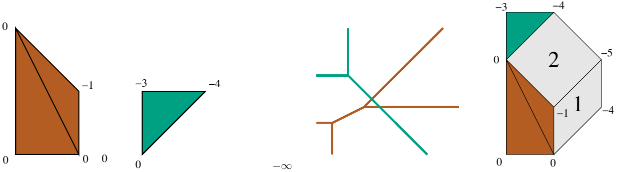
Now change coefficients:



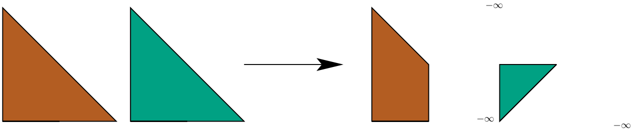
How do we get started?



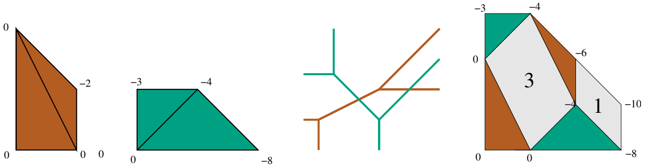
Notice: if height goes to $-\infty$ pieces break off.



This allows us to do a <u>total degree homotopy</u> to get started:



Which lifts give rise to the cell "3"?

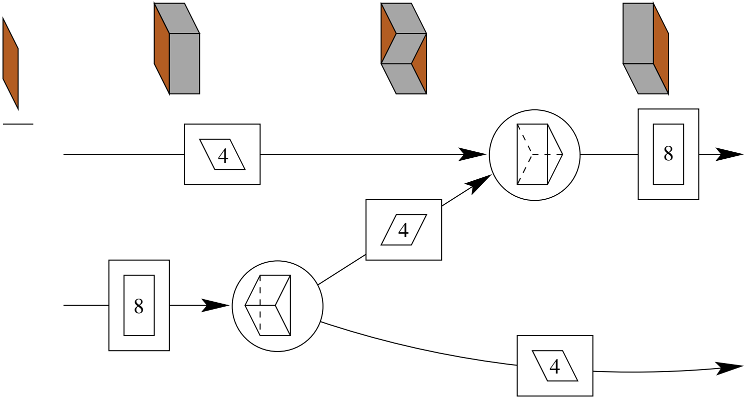


One inequality is $(0, 1, 2, -3, -1, 0, 1, 0) \cdot \omega \geq 0$

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 2 \\ 0 & 2 & 0 & 1 & 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

- One inequality for each "additional" column.
- Inequality set can be updated efficiently as cell changes.

Path tracking. Paths collide.



But can be treated independently with Reverse Search [Avis, Fukuda].

- Split cycle-free directed graph into a forest.
- Compute each tree recursively.

Regeneration (Hauenstein,Sommese,Wampler,2011)

Suppose we want to solve $f_1 = \cdots = f_n = 0$.

- Choose random linear polynomials $l_1, \ldots, l_n$

Now solve

- $l_1 = l_2 = \cdots = l_n = 0$
- $f_1 = l_2 = \cdots = l_n = 0$
- $f_1 = f_2 = \cdots = l_n = 0$
- ...
- $f_1 = f_2 = \cdots = f_n = 0$

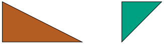
Single step:
Choose $\deg(f_2)$ random linear polynomials $\mathcal{L}_1, \ldots, \mathcal{L}_{\deg(f_2)}$.
Do homotopies:

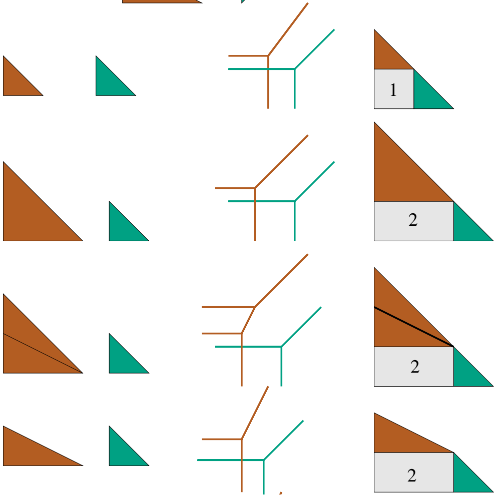
$$\boxed{f_1 = l_2 = \cdots = l_n = 0} \longrightarrow \boxed{f_1 = \mathcal{L}_i = \cdots = l_n = 0} \text{ for } i = 1, \ldots \deg(f_2)$$

$$\boxed{f_1 = \mathcal{L}_1 \cdots \mathcal{L}_{\deg(f_2)} = l_3 = \cdots = l_n = 0} \longrightarrow \boxed{f_1 = f_2 = l_3 = \cdots = l_n = 0}$$

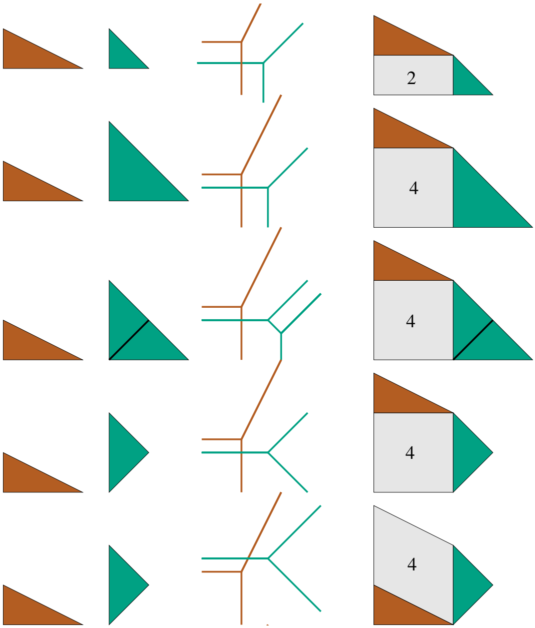
Tropical Regeneration Procedure

Goal: Solve generic system for the two Newton polytopes i.e. find mixed volume of

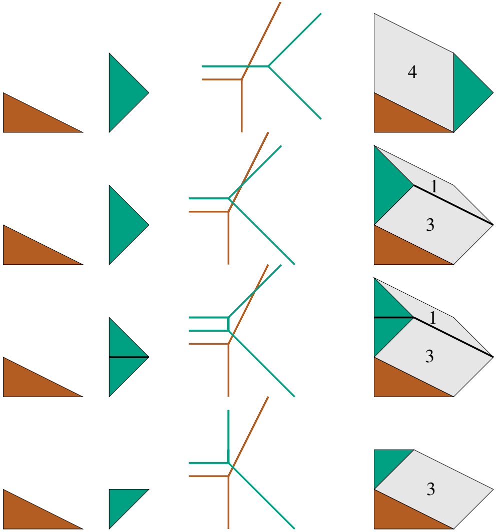




Tropical Regeneration Procedure



Tropical Regeneration Procedure



We have solved a generic system. The mixed volume is 3.

Mixed cell and volume computation (#P hard)

Useful for polyhedral homotopies (Huber-Sturmfels).

2006 Mizutani, Takeda, Kojima: Dynamic enumeration

2011 Lee, Li: Better implementation

2014 Malajovich: First tropical method

2015 Jensen: Tropical homotopy continuation (Gfan)

The new algorithm is <u>Exact</u>, Memory-less and Parallelisable.

Example	n	Mixed vol.	1 thr.	16 thr.*	Mal. (8)	Lee,Li (1)
Cyclic-15	15	35243520	461.3	35.8	4070	36428
Noon-20	20	3486784361	59.0	4.8	6460	1109
Chand.-21	21	1048576	151.6	11.5	7580	1067
Kats.-17	18	131070	4.5	0.5	5310	75619
Eco-22	22	1048576	102.7	8.5	8750	

Timings in seconds. *16 threads - Thanks to Bjarne Knudsen!

Complexity

Malajovich' algorithm (2014) works with "probability 1".

- "Computing mixed volume and all mixed cells in quermassintegral time"

He provides complexity bounds based on geometry.

Similar results hold for the tropical homotopy:

Theorem (Jensen)

The number of edges in $T(A_1, \omega_1) \wedge \cdots \wedge T(A_{n-1}, \omega_{n-1})$ *is*

$$\leq 3 \cdot \mathrm{MixVol}(\mathrm{conv}(A_1), \ldots, \mathrm{conv}(A_{n-1}), \sum_{i=1}^{n-1} \mathrm{conv}(A_i))$$

under the assumption that $\sum_{i=1}^{n-1} \mathrm{conv}(A_i)$ *is full-dimensional and* $\omega_1, \ldots, \omega_{n-1}$ *are generic.*

Are all isolated solutions found?

Algorithm (Tropical homotopy continuation)

Input: $g_1, \ldots, g_n \in \mathbb{R}[x_1, \ldots, x_n]$ *generic start system*
The finite set $T(g_1) \cap \cdots \cap T(g_n) \subseteq \mathbb{R}^n$
(non-generic) $f_1, \ldots, f_n \in \mathbb{R}[x_1, \ldots, x_n]$ *target.*

Output: *All isolated solutions in* $T(f_1) \cap \cdots \cap T(f_n)$

Based on:

Theorem (Osserman and Payne, 2013)

$$\operatorname{codim}(T(h_1) \cap T(h_2)) \leq \operatorname{codim}(T(h_1)) + \operatorname{codim}(T(h_2))$$

However, it follows from complexity results of Theobald that

- It is NP-hard to decide if a solution is isolated.

Future directions: tropical "Smale" problem

Smale's 17th problem:

- "Solving polynomial equations in polynomial time in the average case"

resolved in 2016 (P. Lairez and others).

Tropical version:

- Is there a polynomial time algorithm for finding a single mixed cell?

Ref.: Codenotti, Walther: "Finding a fully mixed cell", 2019

Note: existence of a cell is in P (matroid intersection!).

Earlier results relating tropical and non-tropical complexity:
"Log-barrier interior point methods are not strongly polynomial"
by Allamigeon, Benchimol, Gaubert, Joswig, 2017

References

- Malajovich: “Computing mixed volume and all mixed cells in quermassintegral time” arXiv (2014)
- Jensen: “Tropical homotopy continuation” arXiv (2016)
- Jensen: “An implementation of exact mixed volume computation” ICMS (2016)