Symmetric 3-Mosaics

in collaboration with
Padraig O'Cathain
Oktay Olmez

by

Oliver W. Gnilke
Aalborg University, Denmark

NorCom 2019, August 6, 2019

Designs & Mosaics

Designs

Definition: $t-(v,k,\lambda)$ design

A t-design is an incidence structure on v points and b blocks, such that

i) every block is incident with exactly k points,

ii) any t set of points is contained in exactly λ blocks.

We say a design is simple if all the blocks are pairwise different.

Designs

Definition: $t-(v,k,\lambda)$ design

A t-design is an incidence structure on v points and b blocks, such that

i) every block is incident with exactly k points,
ii) any t set of points is contained in exactly λ blocks.

We say a design is simple if all the blocks are pairwise different.

Theorem

Any $t-(v,k,\lambda)$-design is also an $s-(v,k,\lambda_s)$ design for $s \leq t$.

Example

- We often describe a design by its incidence matrix.

$$\text{Points}\ \overset{\text{Blocks}}{\begin{bmatrix} 1&1&1&1&1&0&0&0&0&0\\ 1&1&0&0&0&1&1&1&0&0\\ 1&0&1&0&0&1&0&0&1&1\\ 0&1&0&1&0&0&1&0&1&1\\ 0&0&1&0&1&0&1&1&1&0\\ 0&0&0&1&1&1&0&1&0&1 \end{bmatrix}} = M$$

$$2-(6,3,2)$$

Complements

- If M is the incidence matrix of an $2-(v,k,\lambda)$, then $\overline{M}=J-M$ is the incidence matrix of an $2-(v,v-k,\overline{\lambda})$ design.

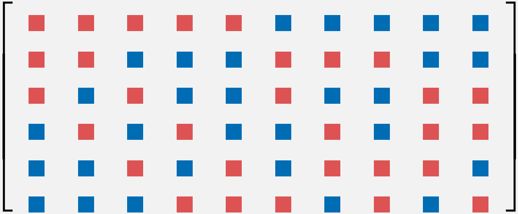
Complements

- If M is the incidence matrix of an $2-(v,k,\lambda)$, then $\overline{M} = J - M$ is the incidence matrix of an $2-(v, v-k, \overline{\lambda})$ design.
- This gives us a two color mosaic.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 & 2 \\ 1 & 2 & 1 & 2 & 2 & 1 & 2 & 2 & 1 & 1 \\ 2 & 1 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 & 1 & 2 & 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 1 & 1 & 1 & 2 & 1 & 2 & 1 \end{bmatrix}$$

Complements

- If M is the incidence matrix of an $2-(v,k,\lambda)$, then $\overline{M} = J - M$ is the incidence matrix of an $2-(v, v-k, \overline{\lambda})$ design.
- This gives us a two color mosaic.



Mosaics

Definition (informal): Mosaic

A c-color mosaic is a collection of c designs with incidence matrices M_i such that

$$\sum_{i=1}^{c} M_i = J.$$

Mosaics

Definition (informal): Mosaic

A c-color mosaic is a collection of c designs with incidence matrices M_i such that

$$\sum_{i=1}^{c} M_i = J.$$

- For $c = 2$ we have mosaics for any parameters for which designs exist.

Mosaics

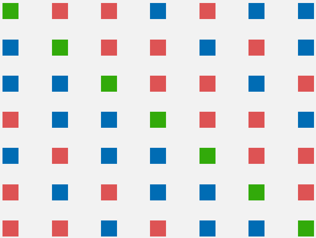
Definition (informal): Mosaic

A c-color mosaic is a collection of c designs with incidence matrices M_i such that

$$\sum_{i=1}^{c} M_i = J.$$

- For $c = 2$ we have mosaics for any parameters for which designs exist.
- Less obvious as soon as $c > 2$.

3-Color Example



$$2-(7,3,1)\oplus 2-(7,3,1)\oplus 2-(7,1,0)$$

Symmetric Mosaics

Symmetric Design

A design that has as many blocks as it has points, $v = b$, is called symmetric.

Symmetric Design

A design that has as many blocks as it has points, $v = b$, is called symmetric.

- Let M be the incidence matrix of a symmetric $2-(v, k, \lambda)$ design, then M^T describes a $2-(v, k, \lambda)$ design.
- In other words, every pair of blocks of a symmetric design intersects in λ points.

Symmetric 3-Mosaics

- For $c = 3$ colors, a mosaic exists iff there exist two designs, such that the sum of their incidence matrices describes a design.

$$M_1 + M_2 = \overline{M_3}$$

Symmetric 3-Mosaics

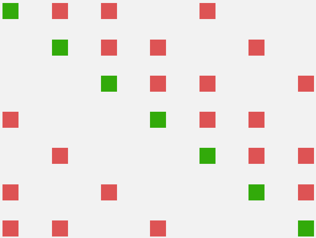
- For $c = 3$ colors, a mosaic exists iff there exist two designs, such that the sum of their incidence matrices describes a design.

$$M_1 + M_2 = \overline{M_3}$$

- Also, $M_1 + M_3 = \overline{M_2}$ and $M_2 + M_3 = \overline{M_1}$

A new parameter

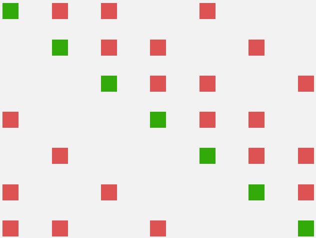
- The blocks of the design $M_1 + M_2$ intersect pairwise in $\overline{\lambda_3} = \lambda_1 + \lambda_2 + \frac{2k_1 k_2}{v-1}$ points



$2-(7,3,1) \oplus 2-(7,1,0) = 2-(7,4,2)$

A new parameter

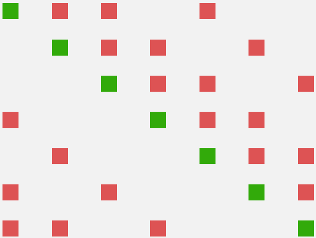
- The blocks of the design $M_1 + M_2$ intersect pairwise in $\overline{\lambda_3} = \lambda_1 + \lambda_2 + \frac{2k_1k_2}{v-1}$ points
- We define $\alpha_{i,j} = \frac{2k_ik_j}{v-1}$



$$2-(7,3,1) \oplus 2-(7,1,0) = 2-(7,4,2)$$

A new parameter

- The blocks of the design $M_1 + M_2$ intersect pairwise in $\overline{\lambda_3} = \lambda_1 + \lambda_2 + \frac{2k_1k_2}{v-1}$ points
- We define $\alpha_{i,j} = \frac{2k_ik_j}{v-1}$



$$2-(7,3,1) \oplus 2-(7,1,0) = 2-(7,4,2)$$

- Here $\alpha_{1,2} = 1$

The case $\alpha_{i,j} = 1$

$\alpha_{i,j} = 1$

If one of the $\alpha_{i,j} = 1$ then the design is of the form $2-(4t-1, 2t-1, t-1) \oplus 2-(4t-1, 2t-1, t-1) \oplus 2-(4t-1, 1, 0)$.

- At least one of the designs has to be the trivial $2-(v, 1, 0)$ design.
  Therefore, there is a design M in the mosaic such that $M + I$ is a design as well. Such designs have been shown to be skew-Hadamard in "Nesting Symmetric Designs", Irish Math. Soc. Bulletin Number 72, Winter 2013, 71–74, P. Ó Catháin

The case $\alpha_{1,2} = \alpha_{1,3}$

$\alpha_{1,2} = \alpha_{1,3}$

If two parameters $\alpha_{1,2} = \alpha_{1,3}$ are identical then the mosaic is skew-Hadamard as well.

- We see that $k_2 = k_3 = k$, and hence we have designs with parameters $2 - (v, k, \frac{k(k-1)}{v-1})$ and $2 - (v, 2k, \frac{2k(2k-1)}{v-1})$.

The case $\alpha_{1,2} = \alpha_{1,3}$

$\alpha_{1,2} = \alpha_{1,3}$

If two parameters $\alpha_{1,2} = \alpha_{1,3}$ are identical then the mosaic is skew-Hadamard as well.

- We see that $k_2 = k_3 = k$, and hence we have designs with parameters $2 - (v, k, \frac{k(k-1)}{v-1})$ and $2 - (v, 2k, \frac{2k(2k-1)}{v-1})$.
- Hence $v - 1$ divides $2k(2k - 1) - 4(k(k - 1)) = 2k$. And it follows that $k = \frac{v-1}{2}$.

The case $\alpha_{1,2} = \alpha_{1,3}$

$\alpha_{1,2} = \alpha_{1,3}$

If two parameters $\alpha_{1,2} = \alpha_{1,3}$ are identical then the mosaic is skew-Hadamard as well.

- We see that $k_2 = k_3 = k$, and hence we have designs with parameters $2-(v, k, \frac{k(k-1)}{v-1})$ and $2-(v, 2k, \frac{2k(2k-1)}{v-1})$.
- Hence $v-1$ divides $2k(2k-1) - 4(k(k-1)) = 2k$. And it follows that $k = \frac{v-1}{2}$.
- Therefore $k_1 = 1$ and the first design is trivial.

Admissible Parameters

$\alpha_{i,j}$	v	k_1	λ_1	k_2	λ_2	
3	211	15	1	21	2	BRC
4	31	6	1	10	3	?
5	43	7	1	15	5	PP 6
5	991	45	2	55	3	
5	1191	35	1	85	6	
6	31	6	1	15	7	?
6	106	15	2	21	4	
6	133	12	1	33	8	
7	43	7	1	21	10	PP 6
7	2731	91	3	105	4	
7	2927	77	2	133	6	
7	3907	63	1	217	12	BRC
8	67	12	2	22	7	
8	91	10	1	36	14	

$\alpha_{i,j}$	v	k_1	λ_1	k_2	λ_2	
8	253	28	3	36	5	
8	381	20	1	76	15	
9	71	15	3	21	6	
9	79	13	2	27	9	
9	111	11	1	45	18	PP 10
9	5815	153	4	171	5	
9	6787	117	2	261	10	
9	9703	99	1	441	20	
10	31	10	3	15	7	?
10	91	10	1	45	22	
10	211	15	1	70	23	
10	496	45	4	55	6	
10	521	40	3	65	8	
10	596	35	2	85	12	
10	871	30	1	145	24	

Bruck–Ryser–Chowla

BRC

Bruck–Ryser–Chowla for designs

If a symmetric $2-(v,k,\lambda)$ design exists, then

$$n = k - \lambda \quad \text{is a square if } v \text{ is even, or}$$

$$X^2 - (k-\lambda)Y^2 = (-1)^{\frac{v-1}{2}}\lambda Z^2 \quad \text{has a non-trivial solution if } v \text{ is odd.}$$

BRC

Bruck–Ryser–Chowla for matrices

If a rational $v \times v$ matrix M exists, such that $MM^T = \lambda J + (k - \lambda)I$, then

$$n = k - \lambda \quad \text{is a square if } v \text{ is even, or}$$

$$X^2 - (k - \lambda)Y^2 = (-1)^{\frac{v-1}{2}} \lambda Z^2 \quad \text{has a non-trivial solution if } v \text{ is odd.}$$

BRC for mosaics (v even)

- Let M_1, M_2 be parts of a mosaic.
- Try $Q = M_1 M_2^T + I$, then

$$\begin{aligned} QQ^T &= (M_1 M_2^T M_2 M_1^T) + M_1 M_2^T + M_2 M_1^T + I \\ &= M_1(n_2 I + \lambda_2 J) M_1^T + \alpha_{1,2}(J - I) + I \\ &= n_2 n_1 I + n_2 \lambda_1 J + \lambda_2 k_1^2 J + \alpha_{1,2}(J - I) + I \\ &= (n_2 \lambda_1 + \lambda_2 k_1^2 + \alpha_{1,2}) J + (n_2 n_1 - \alpha_{1,2} + 1) I \end{aligned}$$

- Now an analogous argument to the classical BRC tells us that if v is even, then

$$n_2 n_1 - \alpha_{1,2} + 1$$

is a perfect square.

Something new

- Let's consider the following hypothetical mosaic on $v = 2380$ points

$$2\text{–}(2380, 183, 14) \oplus 2\text{–}(2380, 793, 264) \oplus 2\text{–}(2380, 1404, 828).$$

Something new

- Let's consider the following hypothetical mosaic on $v = 2380$ points

$$2\!-\!(2380, 183, 14) \oplus 2\!-\!(2380, 793, 264) \oplus 2\!-\!(2380, 1404, 828).$$

- BRC does not exclude these designs, since

$$n_1 = 169 = 13^2, \quad n_2 = 529 = 23^2, \quad n_3 = 576 = 24^2.$$

Something new

- Let's consider the following hypothetical mosaic on $v = 2380$ points

$$2-(2380, 183, 14) \oplus 2-(2380, 793, 264) \oplus 2-(2380, 1404, 828).$$

- BRC does not exclude these designs, since

$$n_1 = 169 = 13^2, \quad n_2 = 529 = 23^2, \quad n_3 = 576 = 24^2.$$

- But the new criterion actually shows that the above mosaic can not exist since

$$n_2 n_1 - \alpha_{1,2} + 1 = 529 \cdot 169 - 122 + 1 = 89280 = 2^6 \cdot 3^2 \cdot 5 \cdot 31$$

is not a perfect square.

Conclusions

Conclusions

- The BRC for mosaics can also be extended to the odd case where it also leads to solving a diophantine equation.

Conclusions

- The BRC for mosaics can also be extended to the odd case where it also leads to solving a diophantine equation.
- The next step is to systematically check which parameter sets are still admissible after applying the new exclusion criteria.

Conclusions

- The BRC for mosaics can also be extended to the odd case where it also leads to solving a diophantine equation.
- The next step is to systematically check which parameter sets are still admissible after applying the new exclusion criteria.
- The smalles open case still seems to be

$$2-(31,6,1)\oplus 2-(31,10,3)\oplus 2-(31,15,7).$$

It stubbornly refuses to give up.

Conclusions

- The BRC for mosaics can also be extended to the odd case where it also leads to solving a diophantine equation.
- The next step is to systematically check which parameter sets are still admissible after applying the new exclusion criteria.
- The smalles open case still seems to be

$$2-(31,6,1)\oplus 2-(31,10,3)\oplus 2-(31,15,7).$$

  It stubbornly refuses to give up.
- We plan on trying to push as much as possible of this machinery into the non-symmetric case.

Thank You!