Latin squares and Latin cubes with forbidden entries

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Completing partial Latin squares

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		3
	2	1

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General problem of completing partial Latin squares is NP-complete.

Avoiding arrays

Avoiding Array Problem. Consider a $n \times n$ array A where every cell contains a subset of the symbols in $\{1, \ldots, n\}$. $A(i, j)$ denotes the set of symbols contained in cell (i, j) of A.

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$$A = \begin{array}{|c|c|c|} \hline 1,2 & & 2 \\ \hline & 2,3 & 1 \\ \hline & 1,2 & 1,3 \\ \hline \end{array}$$

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Thm. There is an $\alpha > 0$, such that for every even n, if P is an $n \times n$ α-sparse PLS, then P is completable. (Value of α approx 10^{-7}.)
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Proof idea.
Use a specific Latin square L with a lot of intercalates.

$L =$

1	2	3	4	5	6
3	1	2	6	4	5
2	3	1	5	6	4
4	6	5	1	3	2
5	4	6	2	1	3
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Swap symbols on many intercalates in a systematic way to form a new Latin square L' from L that agrees with P. □

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Sharp by an example of Wanless (2002).

Avoiding sparse arrays

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Conj. There is a constant $\beta > 0$ such that for every positive integer n, if A is an $n \times n$ $(\beta n, \beta n, \beta n)$-array, then A is avoidable.
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By, an example of Pebody, $\beta \leq 1/3$ in Häggkvist's conjecture.

Completing and avoiding

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The examples by Wanless on non-completable PLS, and by Pebody on non-avoidable arrays, can be combined to prove the following.

Prop. If $\alpha, \beta > 0$ are constants satisfying $\alpha + \beta = 1/3 + \varepsilon$, then there is an α-sparse PLS L, and a $(\beta n, \beta n, \beta n)$-array A, such that there is no completion of L that avoids A.

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Example.

1	2	3
3	1	2
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Def. An $n \times n \times n$ cube is an $(\beta n, \beta n, \beta n, \beta n)$-cube if every cell contains a subset of $\{1, \ldots, n\}$ of size at most βn, and each of the symbols $1, \ldots, n$ occurs at most βn times in every row, column, and file.

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Def. An $n \times n \times n$ cube A is avoidable if there is a Latin cube L such that $L(i,j,k) \notin A(i,j,k)$ for all $1 \leq i, j, k \leq n$.

Avoiding sparse cubes of order 2^t

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Thm. There is a positive constant β such that if $n = 2^t$ and A is a $(\beta n, \beta n, \beta n, \beta n)$-cube of order 2^t, then A is avoidable.
[C.-Markström-Pham 2019]

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[C.-Markström-Pham 2019]

Proof idea. Start from the Boolean Latin Cube L:

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

2	1	4	3
1	2	3	4
4	3	2	1
3	4	1	2

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Permute rows, columns, files, and symbols of A, so that each row, column, file, and symbol class in L contains "few" conflicts, i.e. cells (i, j, k) such that $L(i, j, k) \in A(i, j, k)$.

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Take care of remaining "few conflicts" by swapping symbols on subcubes of order 2; that is, Latin cubes of order 2 contained in L. $\square$.

Thm. There is a positive constant β such that if $n = 2t$ and A is a $(\beta n, \beta n, \beta n, \beta n)$-cube of order $2t$, then A is avoidable.
[C.-Pham 2019+]

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Proof idea. Construct a Latin Cube of even order with many subcubes that we can swap symbols on.
Permute rows, columns, and files of A, so that each row, column and symbol class contains "few" conflicts, i.e. cells (i, j, k) such that $L(i, j, k) \in A(i, j, k)$.

Thm. There is a positive constant β such that if $n = 2t$ and A is a $(\beta n, \beta n, \beta n, \beta n)$-cube of order $2t$, then A is avoidable.
[C.-Pham 2019+]

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Take care of remaining “few conlicts” by swapping symbols on subcubes of order 2. $\square$.

Thank you!