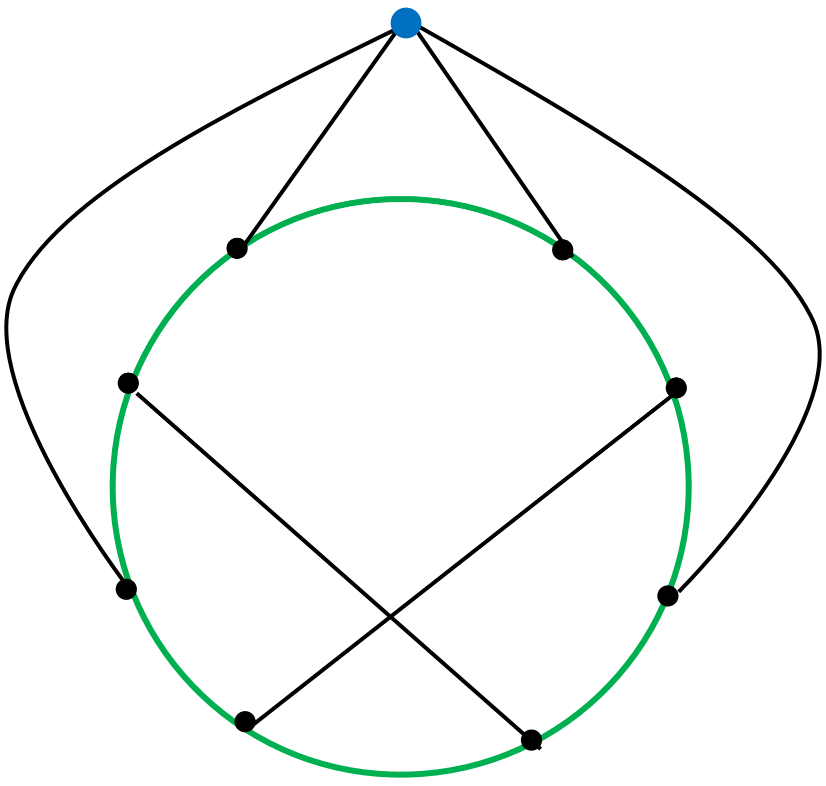
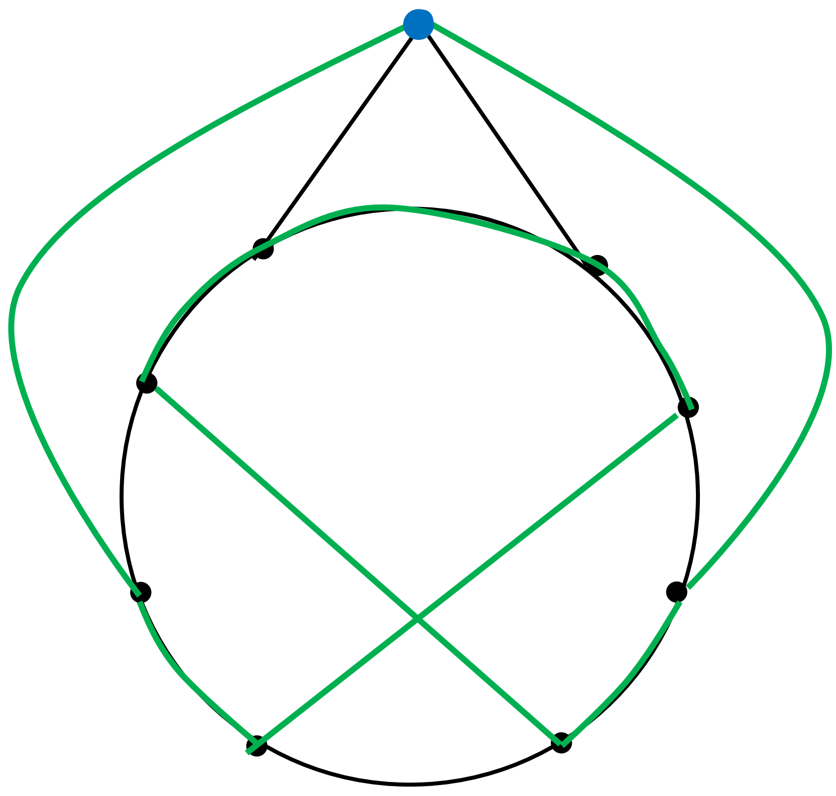
Parity Theorems about Cycles and Trees

Kathie Cameron

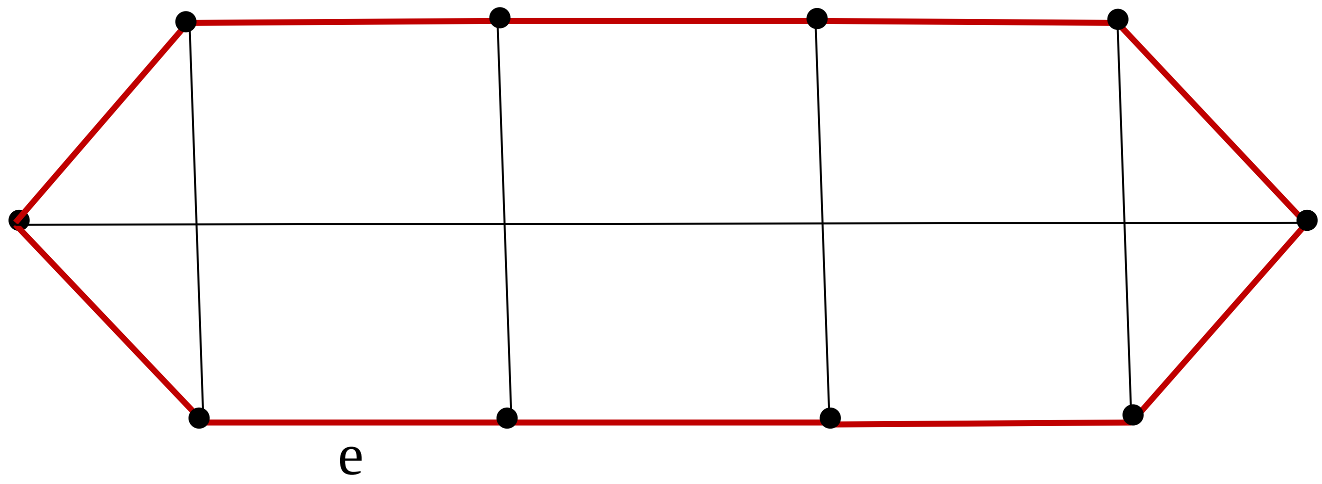
Wilfrid Laurier University

Waterloo, Canada





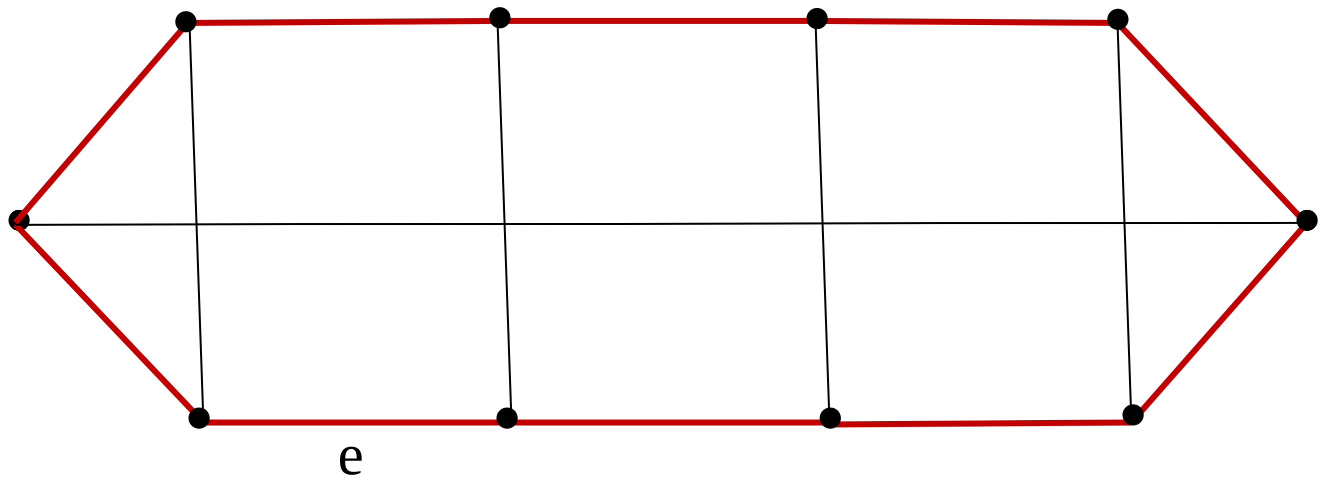
A **Hamiltonian cycle** in a graph G is a cycle containing each vertex of G



Smith's Theorem (Tutte 1946)

Let G be a graph where the degree, d(v), of every vertex v is 3.
Let e be an edge of G.
Then the number of Hamiltonian cycles containing edge e is even.

A **Hamiltonian cycle** in a graph G is a cycle containing each vertex of G



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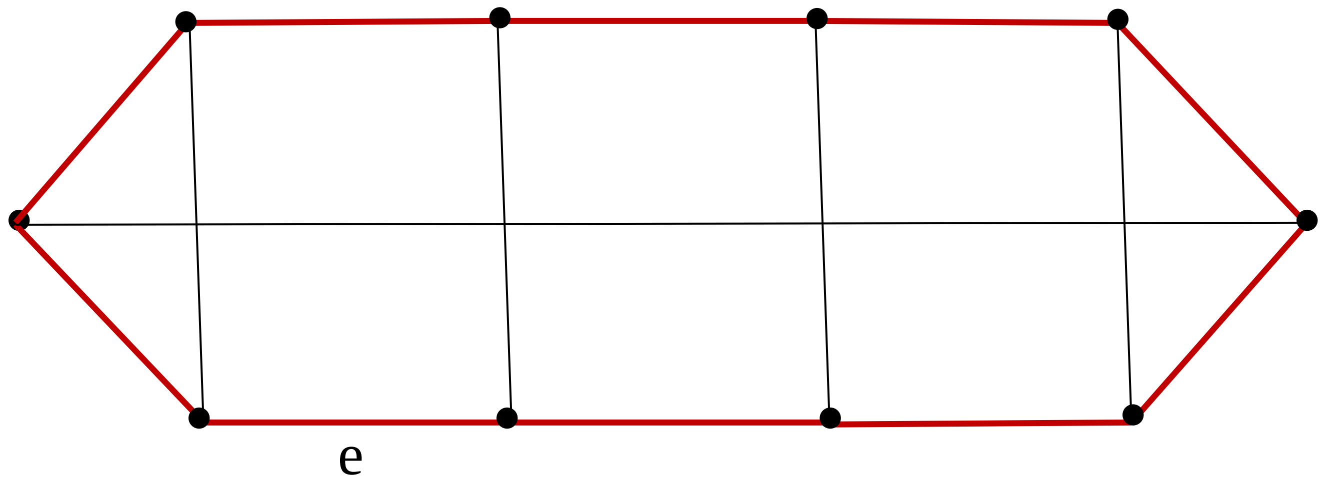

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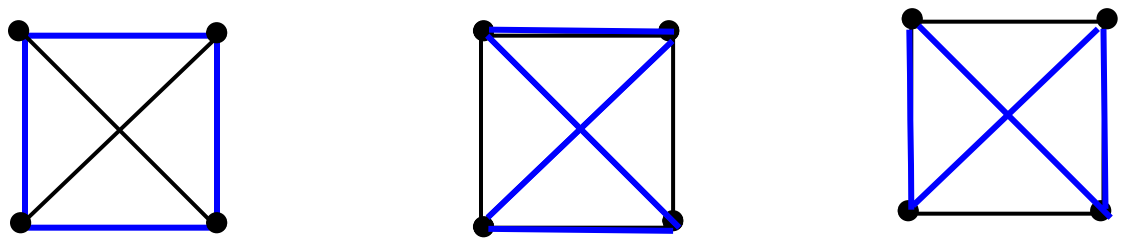
Note that the total number of Hamiltonian cycles may not be even. Eg. K_4



~~Smith's Theorem~~ ~~(Tutte 1946)~~ Theorem (Andrew Thomason 1978)

Let G be a graph where the degree, d(v), of every vertex v is ~~3~~ **odd**.

Let e be an edge of G.

Then the number of Hamiltonian cycles containing edge e is even.

Exchange Graphs

The idea:
For any graph, the number of vertices of odd degree is even.

To prove that the number of **desired structures** is even,
construct a graph **X** such that

desired structures ↔ odd-degree vertices of **X**

Then, given one **desired structure**, to find another **desired structure**, walk in the **exchange graph X** from the given odd-degree vertex to another odd-degree vertex

Theorem (Andrew Thomason 1978)

Let G be a graph where the degree, d(v), of every vertex v is odd.
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Then the number of Hamiltonian cycles containing edge e is even.

Proof.

Hamiltonian path starting with e

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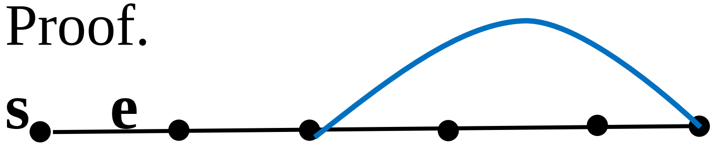

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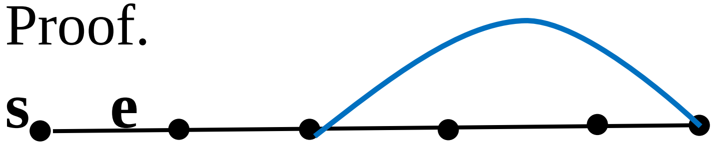

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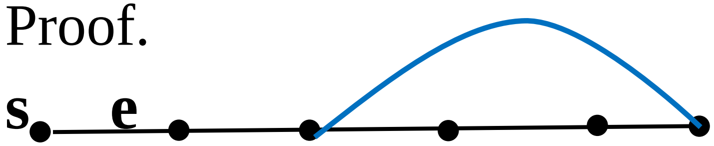

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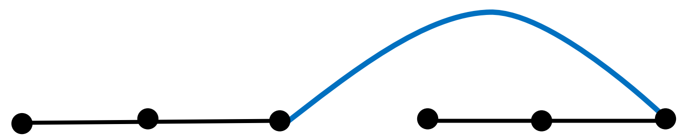

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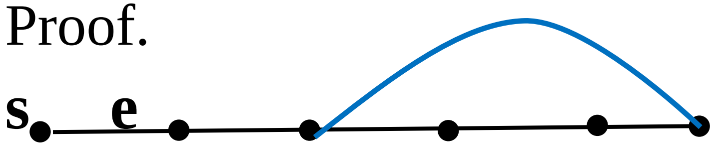

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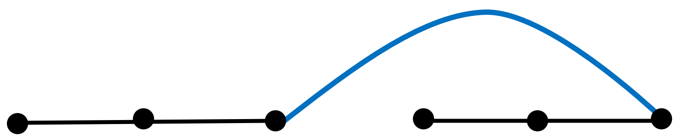
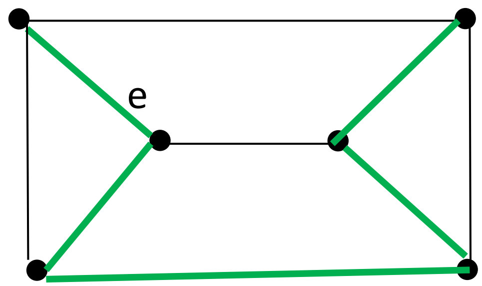


Another Hamiltonian path starting with e

Create the **exchange graph X(G)**:

Vertices are: Hamiltonian paths starting with e

Join two vertices of X(G) if the Hamiltonian paths they correspond to are related by an exchange as above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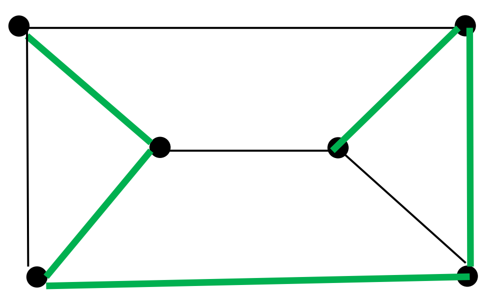

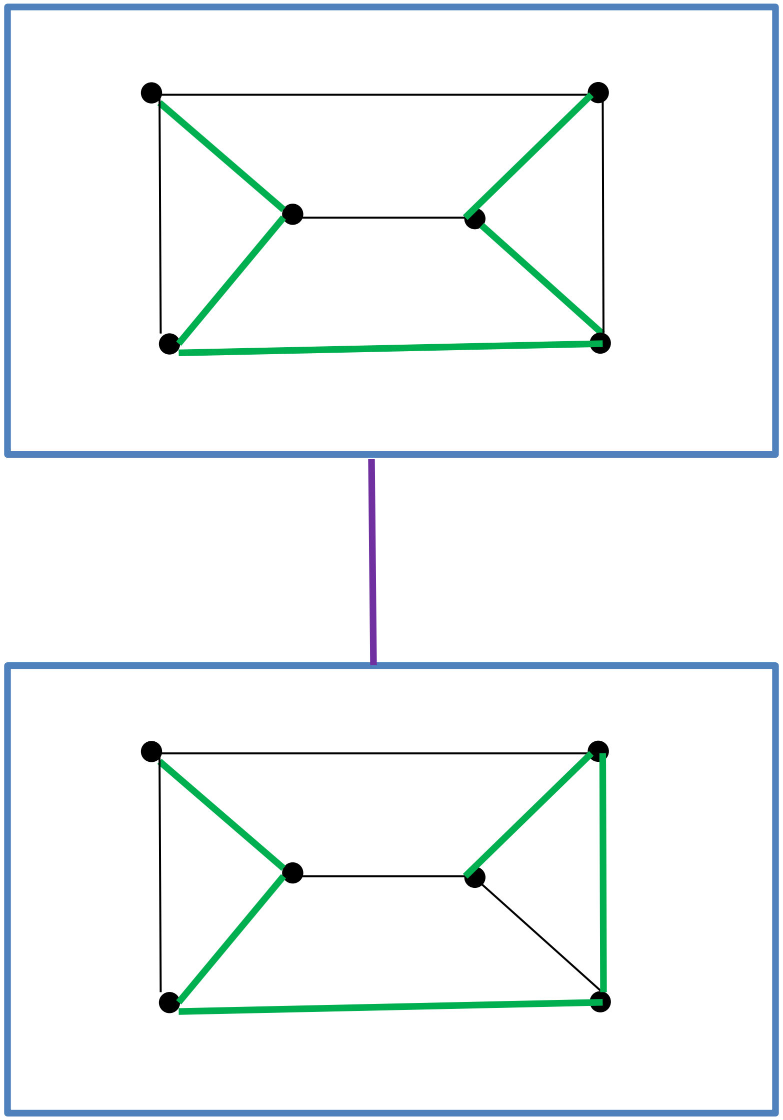

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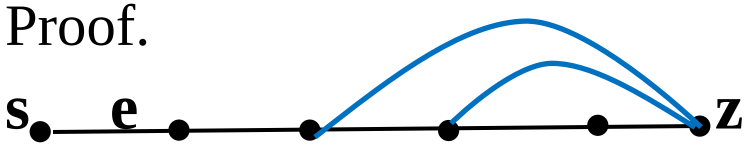

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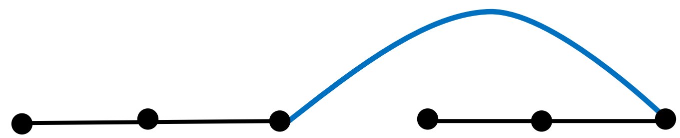
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Degrees in X(G):

Let P be a vertex of X(G), i.e., a Hamiltonian path starting with e and ending at z

$\text{degree}_{X(G)}(P) =$ if P is not extendible to a Hamiltonian cycle

if P is extendible to a Hamiltonian cycle

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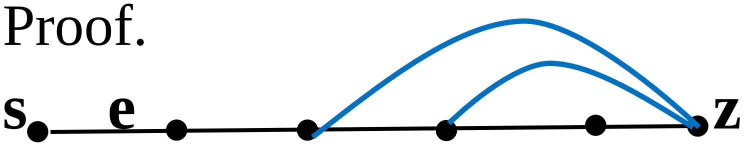

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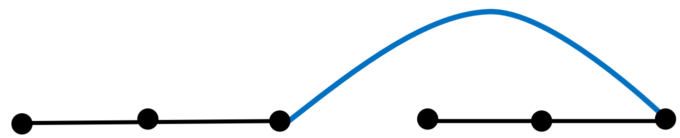
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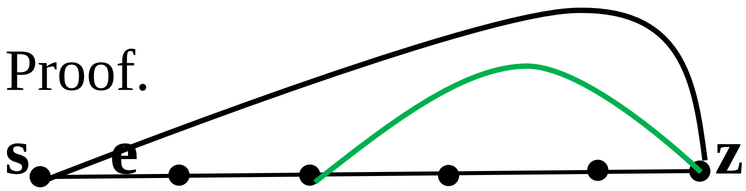

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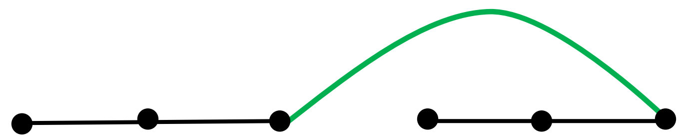

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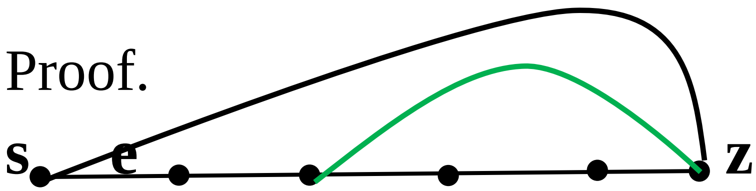

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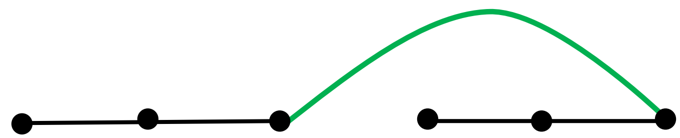

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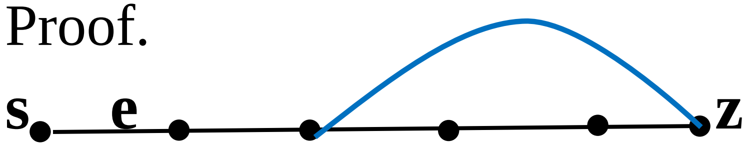

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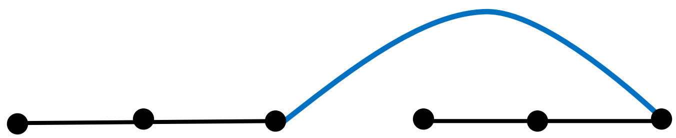

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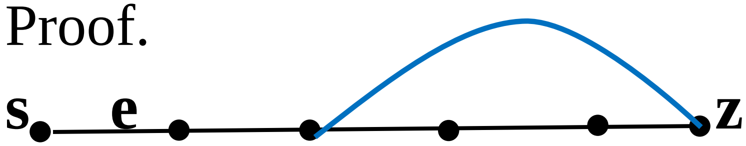

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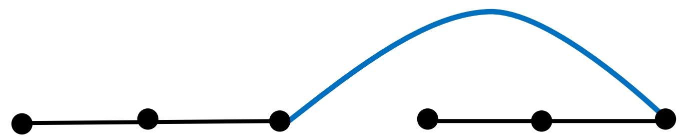

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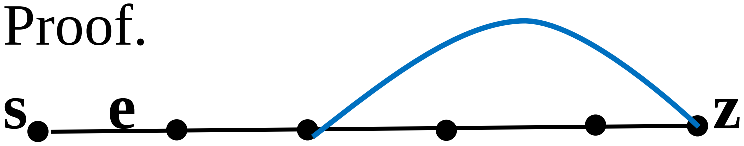

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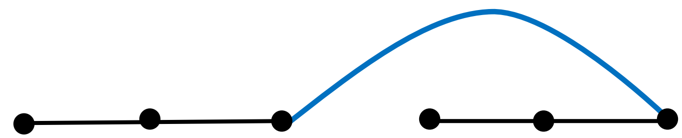

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d(z) - 2 if P is extendible to a Hamiltonian cycle **odd**

In any graph, the number of vertices of odd degree is even.

Thus, the number of Hamiltonian cycles containing e is even. □

Theorem (Shunichi Toida, 1973)
Let G be a graph where the degree, d(v), of every vertex v is even.
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Theorem (Shunichi Toida, 1973)
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Theorem (Andrew Thomason, 1978)
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Since no vertex has odd degree, we could replace **"cycles"** by
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Theorem (Shunichi Toida, 1973)
Let G be a graph where the degree, d(v), of every vertex v is **even**.
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	All vertices have even degree	All vertices have odd degree
#cycles containing e and all odd-degree vertices	odd	even

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	All vertices have even degree	There is an odd-degree vertex	…	All vertices have odd degree
#cycles containing e and all odd-degree vertices	**odd**	**even**	**…**	**even**

Carsten Thomassen and I proved that the number of cycles containing e and all the odd-degree vertices is even as soon as the graph has an odd-degree vertex.

A graph is called ***Eulerian*** if every vertex has even degree.

Theorem (Carsten Thomassen and KC, 2018+)

Let G be a graph and let e be an edge of G.
The number of cycles containing edge e and all the odd-degree vertices
is **odd**
if and only if
G is **Eulerian**.

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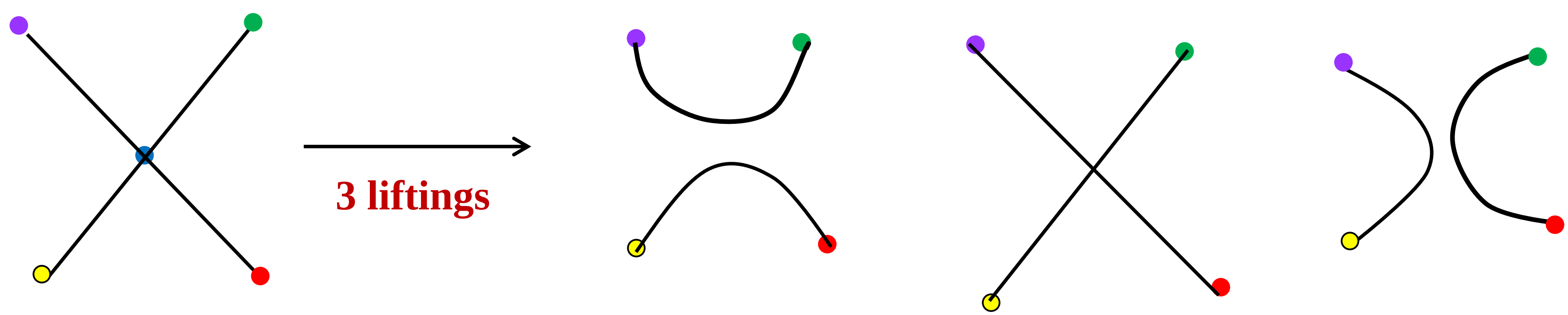
Our proof uses "liftings" of even-degree vertices and is not algorithmic.
I will give an exchange graph proof.



Some background:

Theorem (Andrew Thomason 1978)
Let G be a graph where the degree, d(v), of every vertex v is odd. Let e be an edge of G.
Then the number of Hamiltonian cycles containing edge e is even.

Corollary
Let G be a graph where the degree, d(v), of every vertex v is odd. Let e be an edge of G.
If there is one Hamiltonian cycle containing e, then there is another.

Theorem (Carsten Thomassen, 2016, published 2018)
Let G be a graph where no two even-degree vertices are adjacent.
If there is one cycle containing all the odd-degree vertices, then there is another.

Thomassen used four types of exchange operations,
one of which was Andrew Thomassen's "lollipop" exchange.

Theorem (KC, 2017+)
Let G be a graph where no two even-degree vertices are adjacent. Let e be an edge of G.
Then the number of cycles containing e and all the odd-degree vertices is even.

Follows from
Theorem (KC, 2017+)
Let G be a bipartite graph with bipartition (A,B), where every vertex in A has odd degree and every vertex in B has even degree. Let e be an edge of G.
Then the number of cycles containing e and all the odd-degree vertices is even.

Some background:

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Theorem (KC, 2017+)
Let G be a graph where no two even-degree vertices are adjacent.
Let e be an edge of G.
Then the number of cycles containing e and all the odd-degree vertices is even.

But it turned out that by generalizing the exchange operations into two types, one can get an exchange graph proof of a more general result.

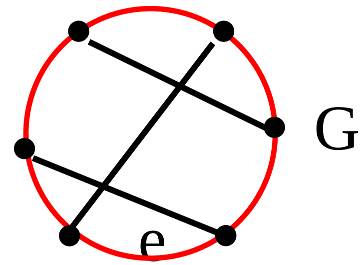
Theorem (Carsten Thomassen and KC, 2018+)
Let G be a graph with an odd-degree vertex and let e be an edge of G.
The number of cycles containing edge e and all the odd-degree vertices is **even**.

This, combined with Toida's Theorem gives:

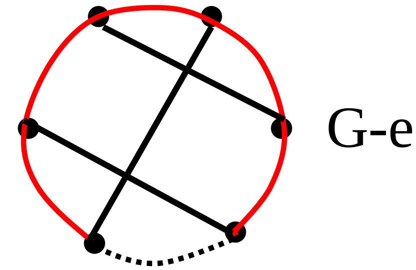
Theorem (Carsten Thomassen and KC, 2018+)
Let G be a graph and let e be an edge of G.
The number of cycles containing edge e and all the odd-degree vertices
is **odd**
if and only if
G is **Eulerian**.

Jack Edmonds and I (1999) gave an exchange graph proof of Toida's Theorem.

A Hamiltonian cycle containing edge e in G



corresponds to a Hamiltonian path in G – e
which is a spanning tree with degree with degree 1
at the ends of e and 2 elsewhere



Defn. Given a graph G and a subgraph S of G (for us, a tree),
the **excess degree** of a vertex is its degree in G – E(S).

Theorem (Ken Berman, 1986)
Suppose graph G has a spanning tree T where the excess degree of each vertex is odd. Then the number of spanning trees of G with the same degree at each vertex as T is even.

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Jack Edmonds and I gave an exchange graph proof (1999) of a generalization – only vertices which are not leaves need to have excess degree

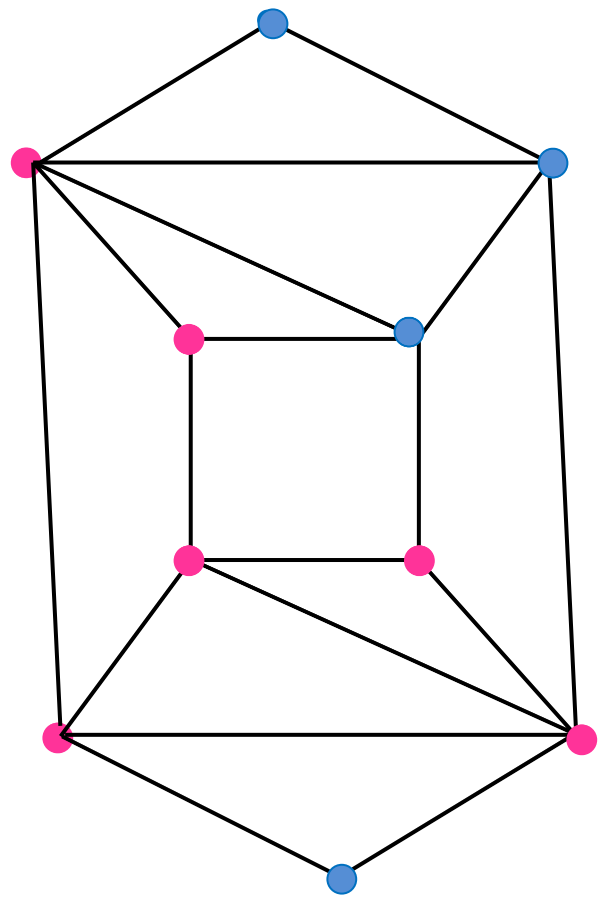
Theorem
Suppose graph G has a spanning tree T where the excess degree of each vertex which is not a leaf of T is odd. Then the number of spanning trees of G with the same degree at each vertex as T is even.

Let **B** be a set of even-degree vertices in a graph G

Let **A = V(G) - B**

A tree **T*** is called ***good*** if it contains all of **A**

and has degree 0 or 2 at each vertex of **B**



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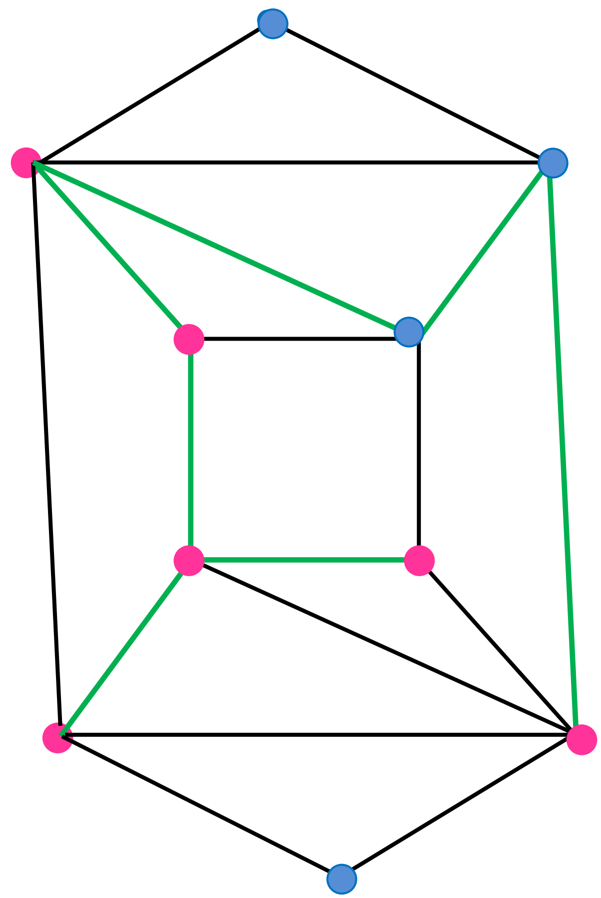

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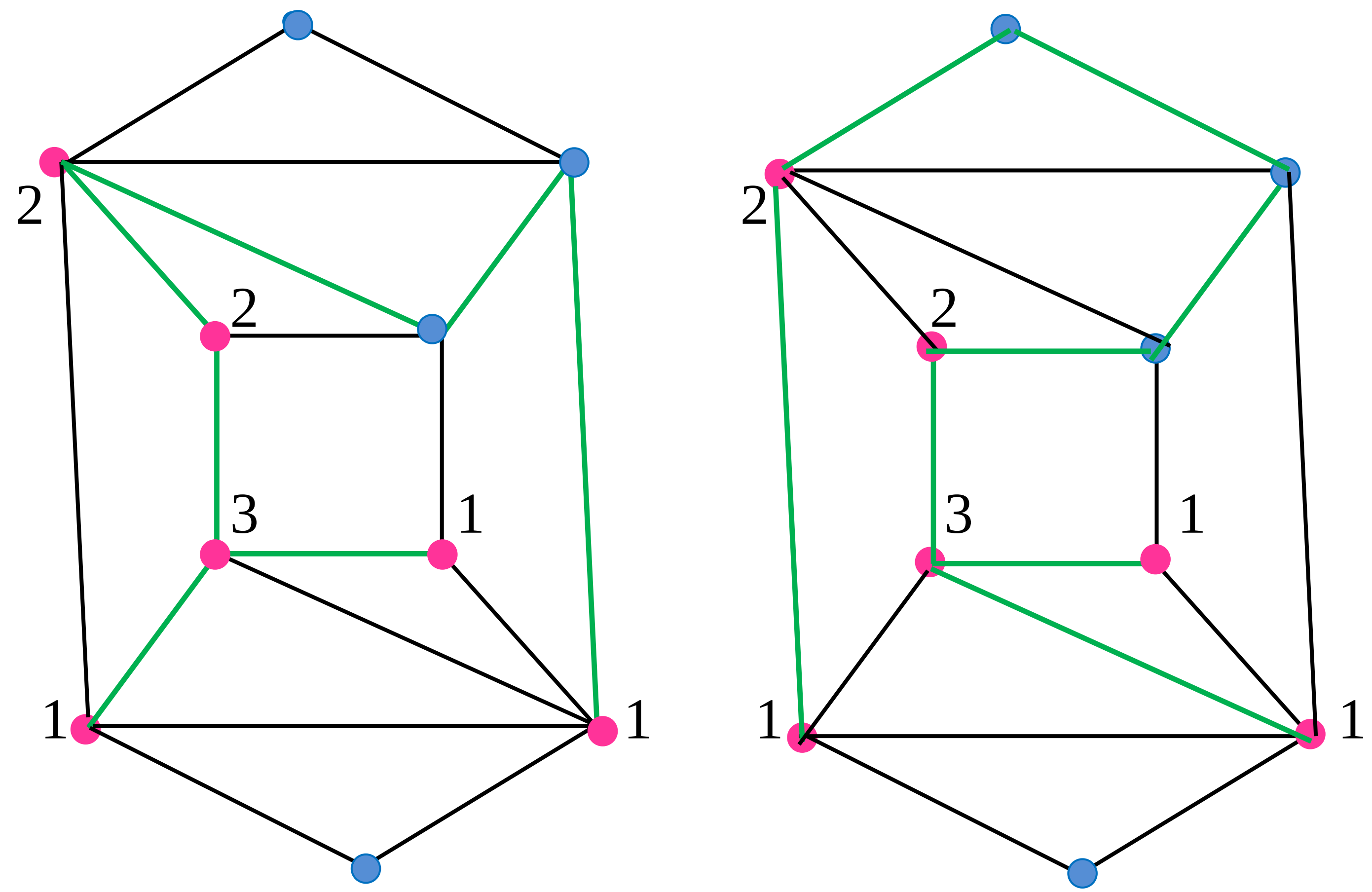

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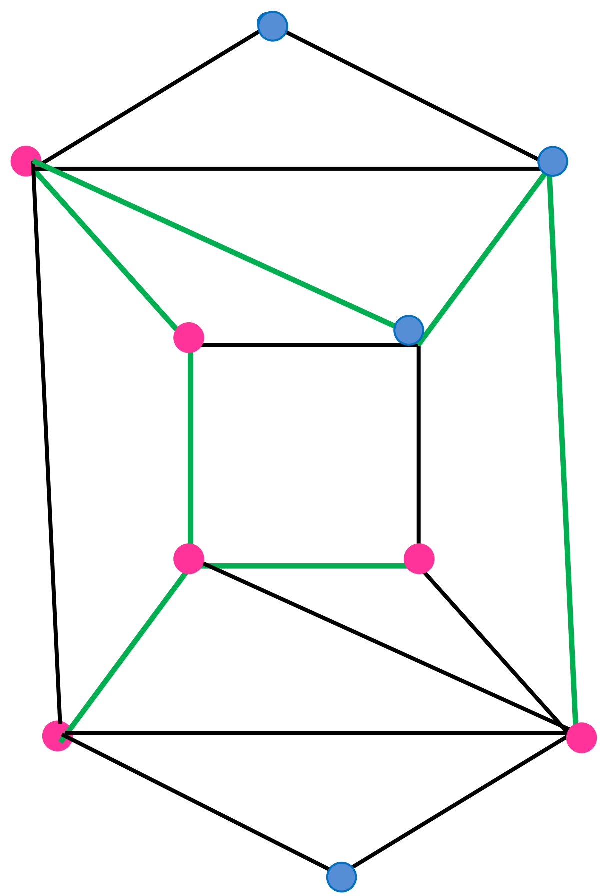
Consider a good tree **T*** such that every vertex of **A** which is not a leaf

of **T*** has odd excess degree



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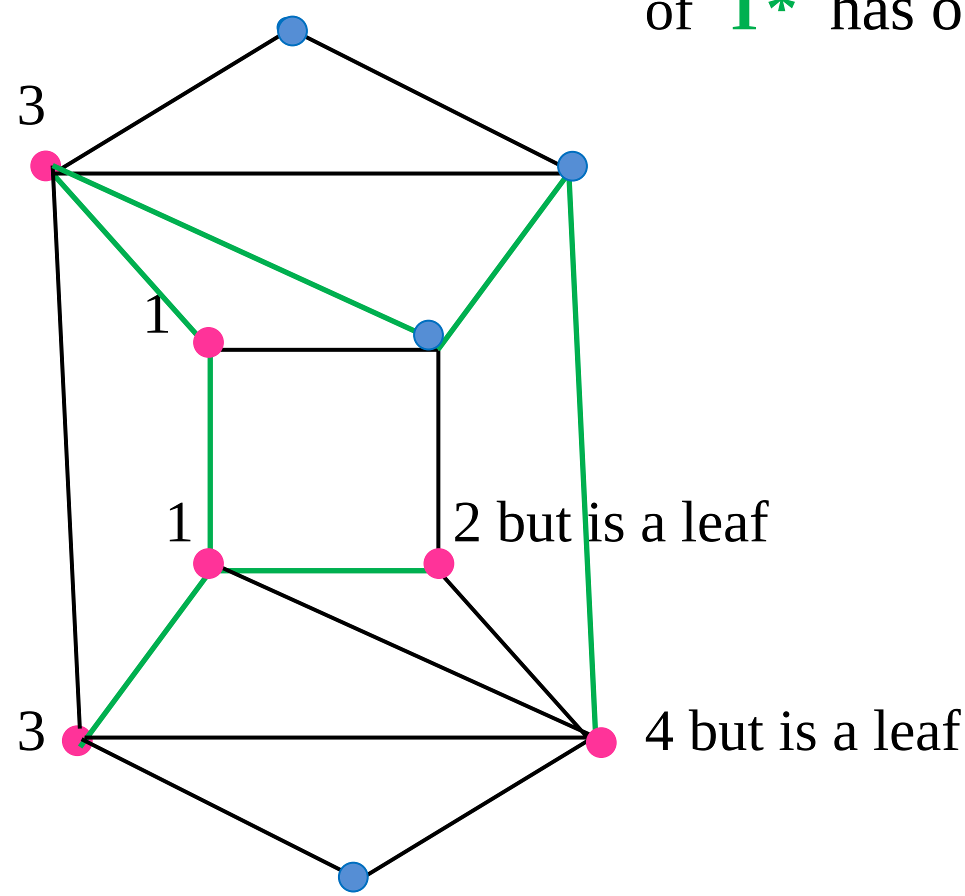

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Theorem (KC 2018+)
Let G be a graph and **B** a set of even-degree vertices in G.
Let **A = V(G) - B.**
Let **T*** be a good tree such that each vertex of **T*** which is not a leaf has odd excess degree.
Then the number of **T***–similar trees is even.

Jack Edmonds and I (2017+) had proved this when G is a bipartite graph with biparition (**A, B**).

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Theorem (Carsten Thomassen and KC, 2018+)

Let G be a graph with an odd-degree vertex and let e be an edge of G.
The number of cycles containing edge e and all the odd-degree vertices is **even**.

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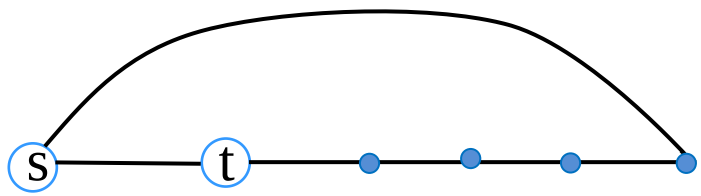
Proof of the cycle theorem from the tree theorem.

Let C be a cycle containing edge e= xy and all the odd-degree vertices.
Let **B** be the set of even-degree vertices in G, other than x and y (which may have odd or even degree). **A = V(G) - B.**
Then **T*** = C-e is a tree in a graph G-e.
For each vertex of **A** in **T*** other than possibly leaves x and y, its excess degree in G-e is (an odd number – 2), thus odd.
By the tree theorem, the number of **T***–similar trees is even.
There is a 1-1 correspondence between these and cycles in G containing e and all the odd-degree vertices.

Theorem **(Shunichi Toida, 1973)**
Let G be a graph where the degree, d(v), of every vertex v is even (i.e. G is Eulerian). Let e = st be an edge of G.
Then the number of cycles containing edge e is odd.

Exchange Graph Proof (Jack Edmonds and KC, 1999)
The odd-degree vertices of the exchange graph X(G) are:

edge e



cycles containing e

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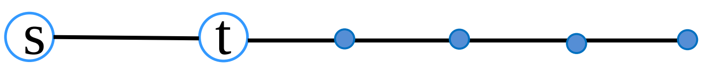
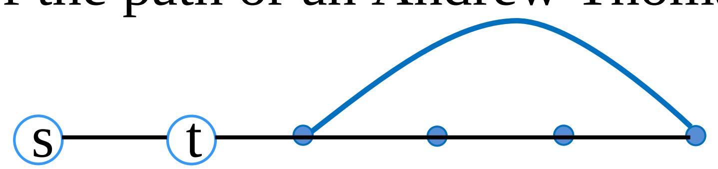
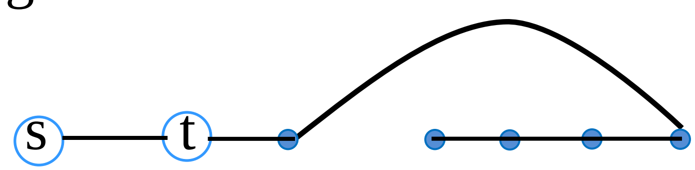


paths beginning with st, of length > 1

The exchange operations are adding or removing an edge meeting the last vertex of the path or an Andrew Thomason exchange:

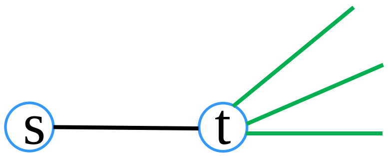


Path starting with e

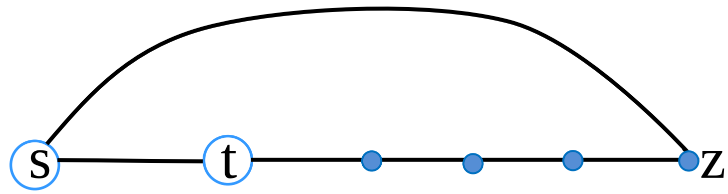


Another path starting with e

The odd-degree vertices of the exchange graph X(G) are:



edge e



cycles containing e

degree is d(t)-1
which is odd

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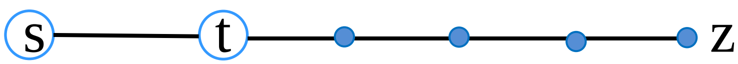

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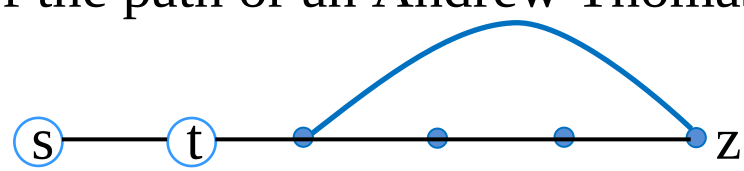

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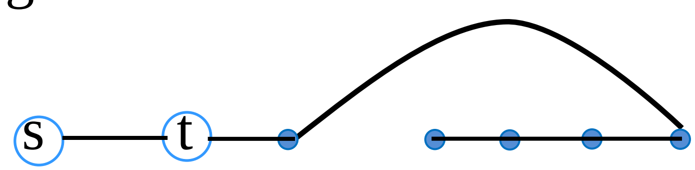

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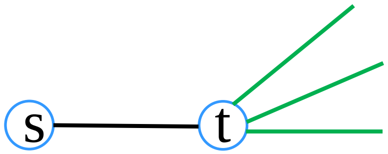

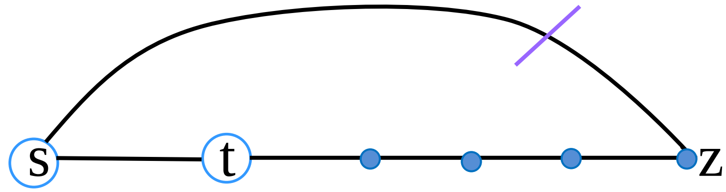

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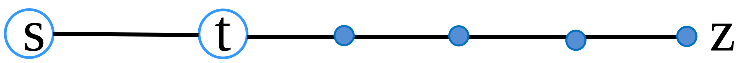

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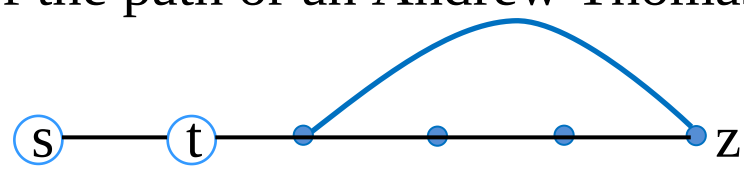
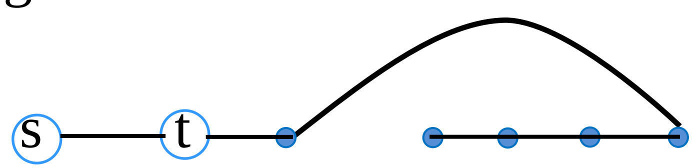


Path starting with e



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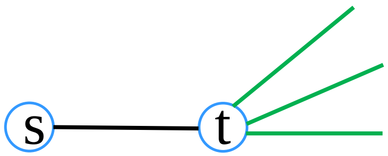
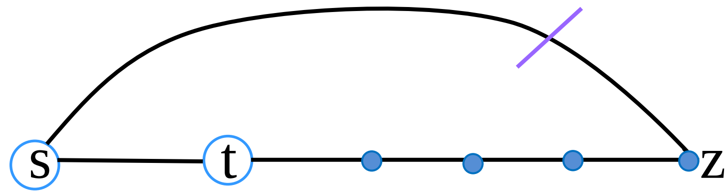


edge e

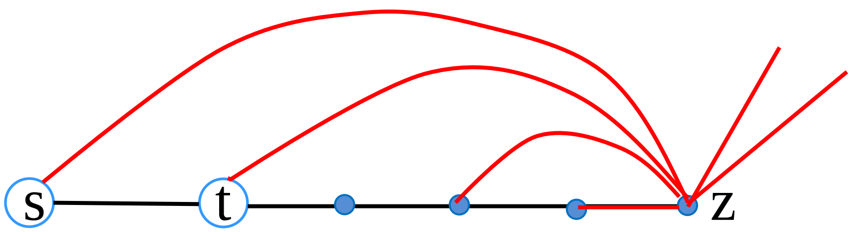
degree is d(t)-1
which is odd



cycles containing e

degree is 1
which is odd

The even-degree vertices of the exchange graph X(G) are paths beginning with st, of length > 1



degree is d(z) which is even

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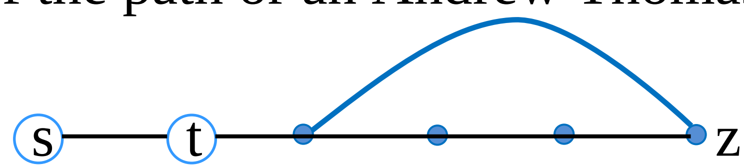

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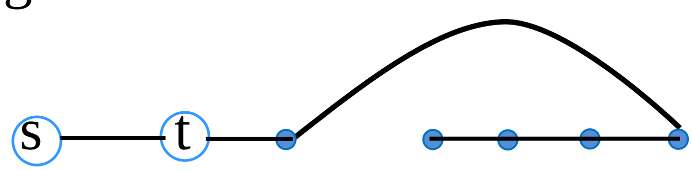

Another path starting with e