Good orientations, antistrong orientations and matroids

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Edge-disjoint spanning trees

Theorem (Tutte)

A graph $G = (V, E)$ has k edge-disjoint spanning trees if and only if, for every partition $\mathcal{F}$ of V, $e_{\mathcal{F}} \geq k(|\mathcal{F}| - 1)$ where $e_{\mathcal{F}}$ is the number of edges with end vertices in different sets of $\mathcal{F}$.

Using matroid techniques, one can obtain a polynomial algorithm which either finds a collection of k edge-disjoint spanning trees of a given graph G or a partition $\mathcal{F}$ for which $e_{\mathcal{F}} < k(|\mathcal{F}| - 1)$ that shows no such collection exists in G.

Branchings

Let $D = (V, A)$ be a digraph and r be a vertex of D. An **out-branching** (respectively, **in-branching)** rooted in r in D is a spanning subdigraph B_r^+ (respectively, B_r^-) of D in which each vertex $v \neq r$ has precisely one arc entering (respectively, leaving) and r has no entering (respectively, leaving) arc.

It follows from the definition that the arc set of an out-branching (respectively, in-branching) of D induces a spanning tree in the underlying graph of D.

D has an out-branching B_r^+ (respectively, an in-branching B_r^-) if and only if there is a directed path from r to v (respectively, from v to r) for every vertex v of D.

Arc-disjoint out-branchings

Generalizing this to arc-disjoint out-branchings:

Theorem (Edmonds 1973)

A digraph $D = (V, A)$ has k arc-disjoint out-branchings, all rooted at the vertex r if and only if $d^-(X) \geq k$ for every $X \subset V - r$

By Menger's theorem, the condition above is equivalent to saying that there are k arc-disjoint (r, v)-paths for every $v \in V - r$

Using flows it can be decided in polynomial time whether a digraph contains k arc-disjoint out-branchings or k arc-disjoint in-branchings.

Lovász found a polynomial algorithm to construct such branchings when they exist

Arc-disjoint in- and out-branchings

Theorem (Thomassen 1986)

It is NP-complete to decide whether a digraph $D = (V, A)$ has an out-branching B_s^+ and an in-branching B_s^- which are arc-disjoint.

Conjecture (Thomassen)

There exits a natural number K such that every K-arc-strong digraph $D = (V, A)$ has a pair of arc-disjoint branchings B_s^+, B_s^- for every $s \in V$.

Conjecture (B-J and Yeo)

There exits a natural number K such that every K-arc-strong digraph $D = (V, A)$ has a pair of arc-disjoint spanning strong subdigraphs.

Acyclic digraphs

Theorem (B.-J,Thomassé, Yeo)

Let $D = (V, A)$ be an acyclic digraph in which s is the unique vertex of in-degree zero and t is the unique vertex of out degree zero. Then D contains a pair of arc-disjoint out-branching and in-branching rooted at s and t respectively if and only if

$$\sum_{x \in X^-} (d^+(x) - 1) \geq |X|. \tag{1}$$

holds for every $X \subseteq V \setminus \{s\}$.

Furthermore, there exists a polynomial algorithm which either finds the desired pair of branchings or a set X which violates (1).

2T-graphs

A **2T-graph** is a graph whose edge set can be decomposed into two edge-disjoint spanning trees.

For a graph $G = (V, E)$ and $X \subseteq V$, the subgraph of G **induced** by X is denoted by $G[X]$.

Theorem (Nash-Williams)

The edge set of a graph G is the union of two forests if and only if

$$|E(G[X])| \leq 2|X| - 2 \ \forall \emptyset \neq X \subseteq V \tag{2}$$

Corollary

A graph $G = (V, E)$ is a 2T-graph if and only if $|V| \geq 2$, $|E| = 2|V| - 2$, and (2) holds.

Generic circuits

Definition

A graph $G = (V, E)$ is called a **generic circuit** if it satisfies the following conditions:

(i) $|E| = 2|V| - 2 > 0$, and

(ii) $|E(G[X])| \leq 2|X| - 3$, for every $X \subset V$ with $2 \leq |X| \leq |V| - 1$.

Generic circuits are important in rigidity theory for graphs.

A celebrated theorem of Laman implies that, for any graph G, the generic circuits are exactly the circuits of the so-called **rigidity matroid** on the edges of G.

Good orientations

Given vertex ordering $\prec$ of a graph G, we use $D_\prec$ to denote the acyclic orientation of G resulting from $\prec$, by orienting all edges forward wrt to $\prec$ and call $\prec$ **good** if $D_\prec$ contains an out-branching and and in-branching which are arc-disjoint. We also call an orientation D of G **good** if $D = D_\prec$ for some good ordering $\prec$ of G.

Thus a graph has a good ordering if and only if it has a good orientation. We call such graphs **good** graphs.

By Theorem 6, one can check in polynomial time whether a given ordering $\prec$ of G is good and return a pair of arc-disjoint branchings in $D_\prec$ if $\prec$ is good.

However, no polynomial time recognition algorithm is known for graphs that have good orderings.

An obvious necessary condition for a graph G to have a good ordering is that G contains a pair of edge-disjoint spanning trees.

This condition alone implies the existence of an orientation D of G having out-branching and an in-branching which are arc-disjoint. But such an orientation may never be made acyclic for certain graphs, which means that G does not have a good ordering.

On the other hand, to certify that a graph has a good ordering, it suffices to exhibit an acyclic orientation of G, often in the form of $D_{\prec}$, and show it contains a pair of arc-disjoint out-branching and in-branching.

Generic circuits are good

Theorem

Let $G = (V, E)$ be a generic circuit, let s, t be distinct vertices of G and let e be an edge incident with at least one of s, t. Then the following holds:

(i) *G has a good ordering $\prec$ with corresponding branchings B^+, B^- in which s is the root of B^+, t is the root of B^- and e belongs to B^+.*

(ii) *G has a good ordering $\prec$ with corresponding branchings B^+, B^- in which s is the root of B^+, t is the root of B^- and e belongs to B^-.*

Theorem

Let G be a 4-regular 4-connected graph in which every edge is on a triangle. Then $G - \{e, f\}$ is a spanning generic circuit for any two disjoint edges e, f. In particular, G admits a good ordering.

Thomassen conjectured that every 4-connected line graph is Hamiltonian; more generally, Matthews and Sumner conjectured that every 4-connected claw-free graph (that is, a graph without $K_{1,3}$ as an induced subgraph) is Hamiltonian. These conjectures are, indeed, equivalent and it suffices to consider 4-connected line graphs of cubic graphs.

The theorem above shows that such graphs have a spanning generic circuit (that is, a spanning cycle in the rigidity matroid).

Generic circuits in 2T-graphs

Every 2T-graph G on two or more vertices contains a generic circuit as an induced subgraph. Indeed, any minimal set X with $|X| \geq 2$ and $|E(G[X])| = 2|X| - 2$ induces a generic circuit in G

We say that H is a **generic circuit of** a graph G if H is a generic circuit and an induced subgraph of G.

Proposition

Let $G = (V, E)$ be a 2T-graph. Suppose that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are distinct generic circuits of G. Then $|V_1 \cap V_2| \leq 1$ and hence $|E_1 \cap E_2| = 0$. In the case when $|V_1 \cap V_2| = 0$, there are at most two edges between G_1 and G_2.

Proposition

Let $G_i = (V_i, E_i)$ where $1 \leq i \leq r$ be the collection of generic circuits of a 2T-graph $G = (V, E)$ and let $\mathcal{G} = (V, \mathcal{E})$ be the hypergraph where $\mathcal{E} = \{V_i : \; 1 \leq i \leq r\}$. Then $\mathcal{G}$ is a hyperforest.

Theorem

There exists a polynomial algorithm $\mathcal{A}$ which given a 2T-graph $G = (V, E)$ as input finds the collection $G_1, G_2, \ldots, G_r$, $r \geq 1$ of generic circuits of G.

Proof: This follows from the fact that the subset system $M = (E, \mathcal{I})$ is a matroid, where $E' \subseteq E$ is in $\mathcal{I}$ precisely when $E' = \emptyset$ or $|E'| \leq 2|V(E')| - 3$ holds, where $V(E')$ is the set of vertices spanned by the edges in E'.

A polynomial independence oracle can be implemented via orientations to achieve bounded indegrees.

The circuits of M are precisely the generic circuits of G. Recall from matroid theory that an element $e \in E$ belongs to a circuit of M precisely when there exists a base of M in $E - e$. Thus we can produce all the circuits by considering each edge $e \in E$ one at a time. If there is a base $B \subset E - e$, then $B \cup \{e\}$ contains a unique circuit C_e which also contains e and we can find C_e in polynomial time by using independence tests in M. Since the generic circuits are edge-disjoint, by Proposition 12, we will find all generic circuits by the process above. $\square$

Corollary

There exists a polynomial algorithm for deciding whether a 2T-graph G is a generic circuit.

Theorem

There exists a polynomial algorithm for deciding whether the vertex set of a 2T-graph $G = (V, E)$ decomposes into vertex disjoint generic circuits. Furthermore, if there is such a decomposition, then it is unique.

The proof above makes heavy use of the structure of generic circuits in 2T-graphs. For general graphs the situation is much worse.

Theorem

It is NP-complete to decide if the vertex set of a graph admits a partition whose members induce generic circuits.

2T-graphs which are disjoint unions of generic circuits

Theorem

Let $G = (V, E)$ be a 2T-graph whose generic components are circuits. If the external edges in G form a matching, then G has a good ordering.

A **double tree** is any graph that one can obtain from a tree T by adding one parallel edge for each edge of T.

The **quotient graph** of a 2T-graph is the graph that we obtain by contracting each generic circuit to a vertex.

Theorem

There exists a polynomial algorithm for checking whether a 2T-graph whose quotient is a double tree has a good ordering.

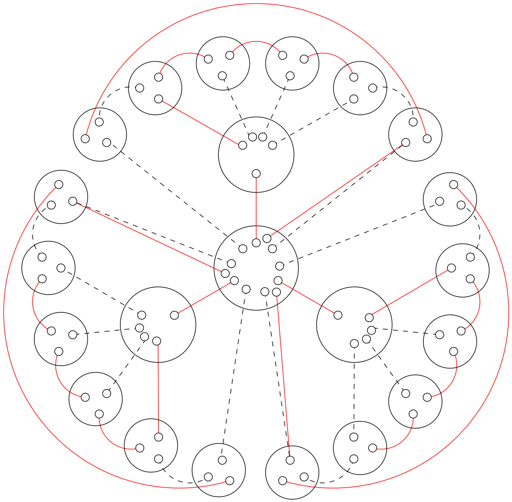


Figure: Example of a 3-connected 2T-graph G such that the set of external edges almost form a matching and G has no good ordering. The solid and dashed edges illustrate two spanning trees along the external edges which can be extended arbitrarily into the circuits.

Remarks on good orientations

Conjecture

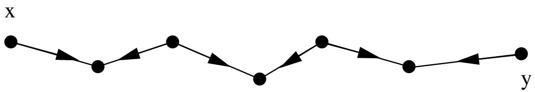
There exists a polynomial algorithm for deciding whether a 2T-graph has a good ordering.

Problem

What is the complexity of deciding whether a given graph has a good ordering?

Part 2: antistrong digraphs and orientations

- In a *digraph* D, an **antidirected path** is a path in which the arcs alternate and beginning and ending with a forward arc.

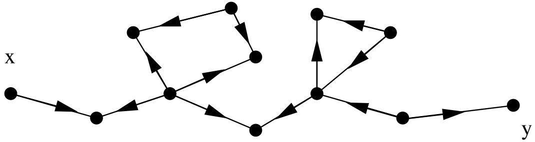


Theorem (A. Yeo, 2014)

Given two vertices x and y of D, it is NP-complete to decide if D admits an antidirected path from x to y.

Antidirected trail

- An **antidirected trail** is a trail (no repeated arc) in which the arcs alternate and beginning and ending with a forward arc.



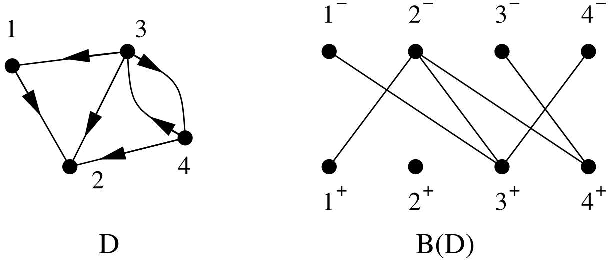
Antidirected trail

- An **antidirected trail** is a trail (no repeated arc) in which the arcs alternate and beginning and ending with a forward arc.

Theorem

It is polynomial to check if there exists an antidirected trail from x to y.

Proof : $B(D)$: the (oriented) adjacency bipartite representation of D.



Antistrong digraph

- A digraph is **antistrong** if for all choices of distinct $x, y \in V(D)$ there exists an andirected trail from x to y.

Theorem

For $|D| \geq 3$, D is antistrong iff $B(D)$ is connected.

- in polytime we can check 'antistrong connectivity'.

Direct results: k-antistrong digraph

- D is k-**antistrong** if for all choices of distinct $x, y \in D$ there exist k-arc-disjoint antidirected trails from x to y.

Theorem

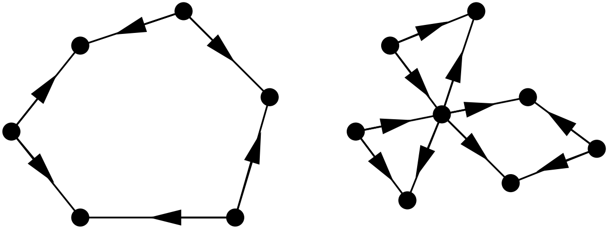
D is k-antistrong iff $B(D)$ is k-edge-connected.

Corollaries:

- In polytime we can check 'k-antistrongness'.
- If D is $2k$-antistrong then D contains k arc-disjoint spanning antistrong subdigraphs.

Direct results: a matroid for antistrong connectivity

- A **CAT** or **closed antidirected trail** is an alternating closed trail.



- The cat-free sets of arcs of D form a matroid M on the arcs of D.
- D is antistrong if and only if M has rank $2|V(D)|-1$

CAT-free orientations

Theorem

Let $G = (V, E)$ *with* $|E| \leq 2|V| - 1$.
G has a CAT-free orientation iff:

$$|E(H)| \leq 2|V(H)| - 1 \quad \text{for all } (\neq \emptyset) \text{ subgraphs } H \text{ of } G \tag{3}$$
$$|E(H)| \leq 2|V(H)| - 2 \quad \text{for all } (\neq \emptyset) \text{ bip. subgraphs } H \text{ of } G \tag{4}$$

Remarks:

- (1) and (2) are necessary.
- No bipartite digraph is antistrong.

Cat-free orientations

Theorem

G satisfies

$$|E(H)| \leq 2|V(H)| - 1 \quad \textit{for all } (\neq \emptyset) \textit{ subgraphs } H \textit{ of } G \qquad (1)$$
$$|E(H)| \leq 2|V(H)| - 2 \quad \textit{for all } (\neq \emptyset) \textit{ bip. subgraphs } H \textit{ of } G \quad (2)$$

iff it can be (edge)-partioned into a forest and an odd pseudoforest

A graph is an **odd pseudoforest** if it contains at most one cycle and if there is a cycle, then it is odd.

Antistrong orientation

In general, for graphs:

Theorem

A graph $G = (V, E)$ has an antistrong orientation if and only if

$$e(\mathcal{Q}) \geq |\mathcal{Q}| - 1 + b(\mathcal{Q}) \qquad (5)$$

for all partitions $\mathcal{Q}$ of V,

where $e(\mathcal{Q})$ denotes the number of edges of G between the different parts of $\mathcal{Q}$ and $b(\mathcal{Q})$ the number of parts of $\mathcal{Q}$ which induce bipartite subgraphs of G.

Corollaries:

- We can decide if a graph admits an antistrong orientation in polytime.
- Every 4-edge-connected nonbipartite graph has an antistrong orientation.
- Every nonbipartite graph with three edge disjoint spanning trees has an antistrong orientation.

Some other results:

- non disconnecting spanning antistrong subdigraph
- connected bipartite 2-detachments
- computing the minimum number of arcs to add to a graph G such that the result is antistrong
- computing the maximum number of arc-disjoint spanning antistrong subdigraphs

Some questions related to antistrongness:

Question: Can we decide in polytime if G has an orientation which is both strong and antistrong?

Question: Suppose D is 1000-arc-strong **and 1000-arc-antistrong**, does D admit two arc-disjoint spanning strong subdigraphs?